The Drinfeld center and topological symmetries

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April 24, 2023

3-dimensional TQFT

- Consider a 3-dimensional (framed) TQFT:

$$\mathbf{Bord}_3^{\mathrm{fr}} \xrightarrow{F} \mathbf{Alg}_1\left(\mathbf{Cat}\right) \ .$$

- The point goes to some monoidal category:

$$F\left(\bullet\right) = \left(\mathcal{C}, *\right) \ .$$

- The interval goes to the identity bimodule:

$$F\left(\ \bullet\!\!-\!\!-\!\!-\!\!-\!\!\bullet\ \right) = {}_{\mathcal{C}}\mathcal{C}_{\mathcal{C}} \ .$$

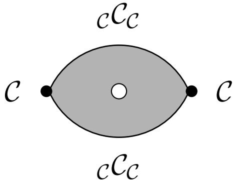
- The circle will be sent to some category:

$$F\left(\bigcirc\right) = ? \in \mathbf{Cat} \cong \mathrm{End}_{\mathbf{Alg}_1(\mathbf{Cat})}\left(1\right) \ .$$

Example of a 3-dimensional TQFT

- Before we identify $F\left(S^1\right)$, let's consider an example.
- Consider the category of vector spaces graded by a finite abelian group G:
$$\mathcal{C} = \mathbf{Vect}\left[G\right] \ . \tag{1}$$
The simple objects are given by 'skyscrapers' $\mathbb{C}_g$ for $g \in G$.
- This has a tensor product given by convolution $*$. On simple objects $\mathbb{C}_g$ it is simply:
$$\mathbb{C}_g * \mathbb{C}_h = \mathbb{C}_{gh} \ . \tag{2}$$
- The TQFT F associated to this particular fusion category is finite gauge theory with gauge group G. If τ is a cocycle for a class in $H^3\left(BG, \mathbb{C}^\times\right)$, we can define a nontrivial associator for $\mathbf{Vect}\left[G\right]$ using τ, resulting in *Dijkgraaf-Witten* theory for $\left(G, \tau\right)$.

The assignment to the circle



- From this picture, we have an action:

$$F\left(\bigcirc\right) \xrightarrow{\sim} \mathrm{End}_{\mathcal{C}\textbf{-bimod}}\left({}_{\mathcal{C}}\mathcal{C}_{\mathcal{C}}\right) .$$

- This map turns out to be an equivalence.[DSS20, Section 3.2.2]
- This has more structure, e.g. a product map given by composition, which we will discuss in a couple of slides.
- But first, let's notice: this is the Drinfeld center!

The Drinfeld center

- The *Drinfeld center* of a tensor category $(\mathcal{C}, *)$ is:

$$\mathcal{Z}\left(\mathcal{C}\right) = \mathrm{End}_{\mathcal{C}\text{-}\mathbf{bimod}}\left({}_{\mathcal{C}}\mathcal{C}_{\mathcal{C}}\right) = \mathrm{End}_{\mathcal{C}\otimes\mathcal{C}^{\mathrm{op}}}\left(\mathcal{C}\right) \ .$$

- So the upshot of the previous slide is:

$$F\left(\ \bigcirc\ \right) \cong \mathcal{Z}\left(F\left(\bullet\right) = \mathcal{C}\right) \ .$$

- The Drinfeld center has a more concrete description: consider the category with objects given by pairs (X, σ_X), where X is an object of $\mathcal{C}$, and σ_x is a natural transformation:

$$\sigma_x \colon X \otimes (-) \to (-) \otimes X \ .$$

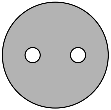
  The morphisms are (appropriately compatible) morphisms in $\mathcal{C}$. See [Eti+15, Prop. 7.13.8] for the equivalence between the two definitions.

Extra structure on $F\left(S^1\right)$

- We have seen that if $F(\bullet) = \mathcal{C}$, then:

$$F\left(\;\bigcirc\;\right) \cong \mathcal{Z}(\mathcal{C}) = \mathrm{End}_{\mathcal{C}\otimes\mathcal{C}^{\mathrm{op}}}(\mathcal{C})\ .$$

- This is naturally a monoidal category: composition of endomorphisms is the same as the multiplication map induced by the pair of pants bordism:

$F\Bigg($  $\Bigg) : F\left(\;\bigcirc\;\right) \otimes F\left(\;\bigcirc\;\right) \to F\left(\;\bigcirc\;\right)\ .$

- In fact there is even a *braiding*, induced by moving one of the “legs” around the other.

Back to the example

- Recall our finite abelian gauge theory example: $F(\bullet) = \mathbf{Vect}[G]$.
- The Drinfeld center of this fusion category turns out to be:

  $$\mathcal{Z}(\mathbf{Vect}[G]) \cong \mathbf{Vect}\left[G \oplus G^{\vee}\right] ,$$

  where $G^{\vee} = \mathrm{Hom}(G, \mathbb{C}^{\times})$ is the character dual.
- The monoidal structure is still convolution, and the braiding is given on simple objects by:

  $$\mathbb{C}_{(g,\chi)} * \mathbb{C}_{(h,\omega)} \xrightarrow{\chi(h)\omega(g)\,\mathrm{id}} \mathbb{C}_{(h,\omega)} * \mathbb{C}_{(g,\chi)} . \tag{3}$$

What does any of this have to do with symmetry?

- A boundary theory $1 \to F$ should be thought of as a "2-dimensional theory with a $\mathcal{C}$-action": usually a 2d theory sends the point to a category, and now it is sent to a $\mathcal{C}$-module category.
- In the finite gauge theory example, where $\mathcal{C} = \mathbf{Vect}\,[G]$, a $\mathcal{C}$-module structure on a category can be thought of as a categorical action of G itself.

Boundary theories for the 3-dimensional theory

- $F : \bullet \mapsto \mathcal{C}$ is the *Turaev-Viro (TV) theory associated to* $\mathcal{C}$.
- The theory sending the circle to a particular braided category $\mathcal{B}$ is the *Reshetikhin-Turaev (RT) theory associated to* $\mathcal{B}$.
- So the RT theory for $\mathcal{Z}(\mathcal{C})$ agrees with the TV theory for $\mathcal{C}$, but not all RT theories may be of this form.

Theorem ([FT21])

An RT theory admits a nonzero boundary theory if and only if it is a TV theory.

4-dimensional theory associated to the Drinfeld center

- The Drinfeld center is a braided category, and turns out to be sufficiently dualizable [BJS21] to define a 4-dimensional TQFT:

$$\alpha \colon \mathbf{Bord}_4 \ni \bullet \mapsto \mathcal{Z}(\mathcal{C}) \in \mathbf{Alg}_2(\mathbf{Cat}) \ .$$

- This is the *Crane-Yetter (CY) theory* associated to the braided category $\mathcal{Z}(\mathcal{C})$.

Upgrading the 3-dimensional theory to a boundary theory

- The Drinfeld center of a tensor category manifestly acts on the original tensor category, since we have a forgetful functor
$$\mathcal{Z}(\mathcal{C}) \to \mathcal{C} \ .$$
- In terms of the theories, this means that F can be upgraded to a boundary condition:
$$\widetilde{F} \colon 1 \to \alpha \ .$$
The value of $\widetilde{F}$ on the point is $\mathcal{C}$ as a $\mathcal{Z}(\mathcal{C})$-module.
- More specifically there is a (α, ρ)-module structure on F, in the sense of [FMT22].

Back to the example

- Consider our running example: if F is G-gauge theory, then the theory

$$\alpha\colon \mathrm{pt} \mapsto \mathcal{Z}\left(\mathbf{Vect}\left[G\right]\right) \cong \mathbf{Vect}\left[G \oplus G^{\vee}\right] \ ,$$

  can be described as the quantization (in the sense of [Fre+10]) of the groupoid $B^2\left(G \oplus G^{\vee}\right)$, twisted by a cocycle for the class

$$\mathrm{ev} \in \mathrm{Hom}\left(G \oplus G^{\vee}, \mathbb{C}^{\times}\right) \cong H^4\left(B^2\left(G \oplus G^{\vee}\right), \mathbb{C}^{\times}\right) \ .$$

- Lagrangian subgroups L of $\left(G \oplus G^{\vee}, \mathrm{ev}\right)$ now give rise to boundary theories $1 \to \alpha$, by quantizing the correspondence (as in [FMT22]):

$$\bullet \leftarrow B^2 L \to B^2\left(G \oplus G^{\vee}\right) \ .$$

- The boundary theories corresponding to $L = G$ and $G^{\vee}$ are related by an “integral transform”.[1]

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[1] This is studied in my upcoming work “A twice-categorified finite Fourier transform”.

The running example as an anomalous theory

- We might wonder if the theory F can be upgraded to have an action of the automorphisms of the center or, more concretely, of the group $\mathrm{O}(G \oplus G^\vee)$.
- At the level of the fusion category itself, this is answered by [ENO10]: this group acts if and only if specific obstructions are trivializable, and an action is determined by a trivialization.
- Passing the obstruction theory from [ENO10] through the "quantization of groupoids" formalism developed in [Fre+10] yields a collection of anomaly theories and symmetry theories for the theory associated to the fusion category we started with.
- This is spelled out in my upcoming work.[2]

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[2] "Equivariance and anomalies of finite topological gauge theory"

Summary:

TV for $\mathcal{C}$:

1. governs $\mathcal{C}$-symmetry,
2. is equivalent to RT for $\mathcal{Z}(\mathcal{C})$, and
3. is (can be upgraded to) a boundary theory for CY for $\mathcal{Z}(\mathcal{C})$.

References

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