Geometric quantization, fusion categories, and Rozansky–Witten theory

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Projective/anomalous TQFTs

Summary:

- Naturally occurring classical symmetry groups typically act *projectively* on the associated quantized theories.
- In many examples this projectivity can be understood as appearing via the homotopy theory of the higher automorphism group of the quantum theory itself.

Context and motivation

- Projective TQFTS in general are relevant to the classification of topological orders: a gapped quantum system is well-approximated at low energy by a *projective* field theory which is topological [Freed].
- The discussion concerning Rozansky-Witten theory is highly relevant to the B-side (spectral side) of the relative Langlands program [Ben-Zvi-Sakellaridis-Venkatesh].
    - Also see [Teleman] and [Braverman-Dhillon-Finkelberg-Raskin-Travkin-Johnson-Freyd].
    - Relatedly, **RW** is the B-side of 3d mirror symmetry. See e.g. [Raskin-Hilburn, Gammage-Hilburn-Mazel-Gee].

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Deformation and geometric quantization

- Classical phase space: symplectic vector space (V, ω), say over $\mathbb{R}$.
  - This has a natural group of symmetries $\mathrm{Sp}(V) = \mathrm{Sp}_{2n}(\mathbb{R})$.
- Deformation quantization (observables): The Weyl algebra:

$$A = T(V) / ([u, v] - \omega(u, v)) = (\mathcal{O}(V), *_\omega) \simeq \mathbb{R}\left[\underline{x}, \underline{p}\right] / ([x_i, p_i] - 1)$$

  - The group $\mathrm{Sp}_{2n}(\mathbb{R})$ still acts on A.
- Geometric quantization (states): The Weil representation:

$$\mathcal{H} = L^2(\ell)$$

where we have chosen a polarization $V \simeq \ell \oplus \ell^\vee$.

  - The group $\mathrm{Sp}_{2n}(\mathbb{R})$ only acts *projectively* on $\mathcal{H}$. Equivalently, a central extension, classified by $w_2 \in H^2(B\,\mathrm{Sp}_{2n}(\mathbb{R}))$, acts linearly on $\mathcal{H}$:

$$\mathbb{Z}/2 \hookrightarrow \mathrm{Mp} \twoheadrightarrow \mathrm{Sp}_{2n}(\mathbb{R})$$

Table of analogies

$d = 1$ (QM)	(V, ω)	$\mathcal{O}(V), *_\omega$	$L^2(\ell)$	$w_2 \in H^2(B\,\mathrm{Sp}_{2n}(\mathbb{R}))$
$d = 3$ (TV)				
$d = 3$ (RW)				

Families

Let X be a (π-finite) groupoid (space). Given a field theory F_0, one can ask if background fields can be inserted, i.e. if there is a family of theories over X with fiber F_0:

$$\begin{array}{ccc} \mathbf{Bord}_d^X & & \\ \uparrow & \overset{F}{\searrow} & \\ \mathbf{Bord}_d & \xrightarrow[F_0]{} & \mathcal{T} \end{array}$$

Two equivalent ways of encoding this notion:

$F\colon \mathbf{Bord}_d^X \to \mathcal{T}$	$F\colon 1 \to \sigma_X^{d+1}$

- Notation: The $(d+1)$-dimensional gauge theory associated to X as in [Freed-Moore-Teleman] is $\sigma_X^{d+1}\colon \mathbf{Bord}_{d+1} \to \mathbf{Alg}(\mathcal{T})$.
- The RHS is an (op)lax natural transformation as in [Johnson-Freyd-Scheimbauer].

Example: $X = BG$

- For G a finite group, families of theories over the groupoid $X = BG$ are equivalent to theories with an action of G.
- For example, fix $d = 1$. Functors

$$F \colon \mathbf{Bord}_1^{BG} \to \mathcal{T} = \mathbf{Vect}$$

  are classified by functors $BG \to \mathbf{Vect}$, which are representations of G.
- On the other hand, the theory σ_{BG}^2 is a functor valued in the Morita category of algebras:

$$\mathbf{Bord}_2 \xrightarrow{\sigma_{BG}^2} \mathbf{Alg}$$

$$* \longmapsto (\mathbb{C}[G], *)$$

- A theory defined relative to the theory σ_{BG}^2 is a morphism in the Morita category from $\mathbb{C}$ to the group algebra, i.e. a module over the group algebra.

Anomalies

Let X be a (π-finite) groupoid (space), and now consider *projectivity data on* X:

$$\begin{array}{ccc} B^d\mathbb{C}^\times & \longrightarrow & \widetilde{X} \\ & & \downarrow \\ & & X \xrightarrow{\alpha} B^{d+1}\mathbb{C}^\times \end{array}$$

Projective/anomalous d-dimensional TQFT with background X-fields:

Theorem (VD23)

TFAE:

$F\colon 1_X \to \alpha$	$F\colon 1 \to \sigma_{X,c}$
$F\colon \mathbf{Bord}_d^{\widetilde{X}} \to \mathcal{T}$	$F\colon 1 \to \sigma_{\widetilde{X}}$

E.g. $X = BG$, $\widetilde{X} = B\widetilde{G}$ for $\widetilde{G}$ a central extension of G classified by α.

projective rep	mod over the twisted gp alg
rep of the ext'n	mod over gp alg of the ext'n

Anomalies

- We will focus on the analogue of a module over the twisted group algebra:

$$\sigma_{X,c}^{d+1} \colon \mathbf{Bord}_{d+1} \to \mathbf{Alg}\left(\mathcal{T}\right) \qquad\qquad F \colon 1 \to \sigma_{X,c}$$

- Caveat: For some of the examples we will consider, the theory $\sigma_{X,c}^{d+1}$ has not been formally constructed.
- In these cases, one can consider the analogue of a projective representation instead: $1 \to \alpha$
  where everything is rigorous. See [VD23, Hypothesis Q] for more details.

Recasting quantization

- The theory of states defines a 1-dimensional QFT **GQ**.
- The classical symmetry group $G = \mathrm{Sp}_{2n}(\mathbb{R})$ acting projectively on $\mathcal{H}$ is equivalent to the theory living relative to twisted G-gauge theory:

$$\begin{array}{ccccc} \mathrm{U}(1) & \longrightarrow & \mathrm{U}(\mathcal{H}) & \longrightarrow & \mathrm{U}(\mathcal{H})/\mathrm{U}(1) \\ \uparrow & & \uparrow & & \uparrow \\ \mathbb{Z}/2 & \longrightarrow & \mathrm{Mp} & \longrightarrow & \mathrm{Sp} \end{array}$$

$$\begin{array}{ccc} B\,\mathrm{Mp} & & \\ \downarrow & & \\ B\,\mathrm{Sp} & \xrightarrow{w_2} & B^2\mathbb{Z}/2 \end{array}$$

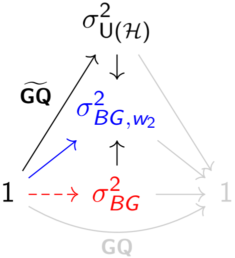
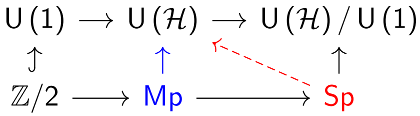
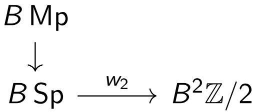
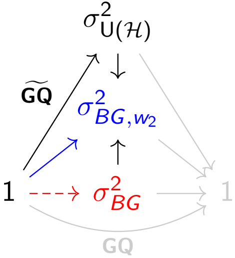


Table of analogies

$d = 1$ (QM)	(V, ω)	$\mathcal{O}(V), *_\omega$	$L^2(\ell)$	$w_2 \in H^2(B\,\mathrm{Sp}_{2n}(\mathbb{R}))$
$d = 3$ (TV)				
$d = 3$ (RW)				







Fusion categorical analogue of deformation quantization

- Analogue of classical phase space: finite abelian group Λ equipped with a quadratic form $q \colon \Lambda \to \mathbb{C}^\times$.
  - Has a natural group of symmetries $\mathrm{O}(\Lambda, q)$.
- Analogue of deformation quantization: braided fusion category

$$(\mathbf{Vect}[\Lambda], *, \beta_q) \qquad \beta_{\mathbb{C}_\ell, \mathbb{C}_k} \colon \mathbb{C}_\ell * \mathbb{C}_k \xrightarrow{\langle \ell, k \rangle_q \mathrm{id}} \mathbb{C}_k * \mathbb{C}_\ell \,.$$

  - Retains an action of $\mathrm{O}(\Lambda)$.
- Analogue of geometric quantization: fusion category

$$\mathbf{Vect}[L]$$

where we have chosen $\Lambda \simeq L \oplus L^\vee$ such that $q = \mathrm{ev}$.
  - The group $\mathrm{O}(\Lambda)$ only acts projectively on $\mathbf{Vect}[L]$.

The fusion categorical anomaly

Recasting the previous slide in terms of TQFTs:

- There is a (framed, fully-extended) Turaev-Viro theory, which sends the point to $\mathbf{Vect}[L]$, in the Morita 3-category of fusion categories [Douglas-Schommer-Pries-Snyder].
- Using the obstruction theory of [Etingof-Nikschych-Ostrik]:

Theorem (VD23)

The Turaev-Viro theory for $\mathbf{Vect}[L]$ *can be upgraded to a theory defined relative to a twisted gauge theory for* $\mathrm{O}(\Lambda)$*:*

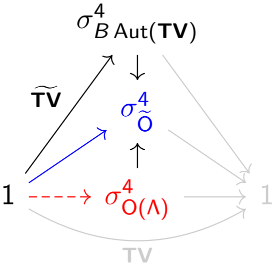
$$\mathbf{TV}: 1 \to \sigma_{B\,\mathrm{O}(\Lambda),c}$$

Table of analogies (reprise)

$d = 1$ (QM)	(V, ω)	$\mathcal{O}(V), *_\omega$	$L^2(\ell)$	$w_2 \in H^2(B\,\mathrm{Sp}_{2n}(\mathbb{R}))$
$d = 3$ (TV)	(Λ, q)	$\mathbf{Vect}[\Lambda], \beta_q$	$\mathbf{Vect}[L]$	$O_4 \in H^4(B\,\mathrm{O}(\Lambda))$
$d = 3$ (RW)				

$$\begin{array}{ccccc} B^2\mathbb{C}^\times & \longrightarrow & \mathrm{Aut}(\mathbf{TV}) & \longrightarrow & \pi_{\leq 1}\mathrm{Aut}(\mathbf{TV}) \\ \| & & \uparrow & \nwarrow & \uparrow \\ B^2\mathbb{C}^\times & \longrightarrow & \widetilde{O} & \longrightarrow & \mathrm{O}(\Lambda) \end{array}$$

$$\begin{array}{ccc} B\widetilde{O} & & \\ \downarrow & & \\ B\,\mathrm{O}(V) & \xrightarrow{O_4} & B^4\mathbb{C}^\times \end{array}$$



Shifted deformation quantization

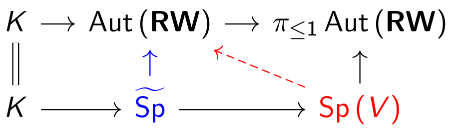
- The classical phase space: 2-shifted symplectic stack (M, ω) as in [Calaque, Pantev, Toën, Vaquié, Vezzosi].
- The deformation quantization can be thought of as a braided deformation of the derived category $\mathbf{QC}(M)$ of quasi-coherent sheaves.
    - Think: the deformation quantization of k-shifted is an $\mathbb{E}_{k+1}$-algebra, modules over this form an $\mathbb{E}_k$-category.
- This will be the assignment of the Rozansky-Witten theory to the circle. [Roberts-Willerton]
    - Compare: The braided fusion category $\mathbf{Vect}[\Lambda]$ was assigned to the circle by Tuaev-Viro.

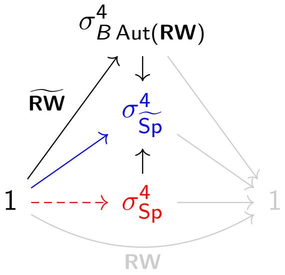
Rozansky-Witten theory

- The geometric quantization of a shifted symplectic stack is less well-studied, but a theory is developed in [Safronov].
- The explicit construction of this geometric quantization again typically appeals to a polarization, and therefore has more subtle equivariance properties than the deformation quantization.
- The output should be a certain 2-category, which is closely related to existing constructions of Rozansky-Witten theory. [Rozansky-Witten, Roberts-Willerton, Kapustin-Rozansky-Saulina, Brunner-Carqueville-Fragkos-Roggenkamp, Gammage-Hilburn-Mazel-Gee]
- The theory should fit into the framework of the AKSZ construction. [Alexandrov-Kontsevich-Schwarz-Zaboronsky, Qiu-Zabzine, Scheimbauer-Calaque-Haugseng, Stefanich, Riva]

Table of analogies (reprise[2])

$d=1$ (QM)	(V,ω)	$\mathcal{O}(V), *_\omega$	$L^2(\ell)$	$w_2 \in H^2(B\,\mathrm{Sp}_{2n}(\mathbb{R}))$
$d=3$ (TV)	(Λ, q)	$\mathbf{Vect}[\Lambda], \beta_q$	$\mathbf{Vect}[L]$	$O_4 \in H^4(B\,\mathrm{O}(\Lambda))$
$d=3$ (RW)	(M,ω)	$\mathbf{QC}(M), \beta_\omega$	$\mathbf{RW}(*)$	$? \in H^4(B\,\mathrm{Sp}(M))$





My current work in this direction

- The analogue of Crane-Yetter in the **RW** context.
  - One strategy for accessing this projective Sp-action.
- General relationship between the prequantum k-gerbe and the anomaly $(k+1)$-gerbe.
  - One strategy for understanding the impact of changing the polarization on the shifted geometric quantization.

Thank you!