PROJECTIVE SYMMETRIES OF THREE-DIMENSIONAL TQFTS

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Abstract. Quantum field theory has various projective characteristics which are captured by what are called anomalies. This paper explores this idea in the context of fully-extended three-dimensional topological quantum field theories (TQFTs).

Given a three-dimensional TQFT (valued in the Morita 3-category of fusion categories), the anomaly identified herein is an obstruction to gauging a naturally occurring orthogonal group of symmetries. In other words, the classical symmetry group almost acts: There is a lack of coherence at the top level. This lack of coherence is captured by a "higher (central) extension" of the orthogonal group, obtained via a modification of the obstruction theory of Etingof-Nikshych-Ostrik-Meir [ENO10]. This extension tautologically acts on the given TQFT/fusion category, and this precisely classifies a *projective* (equivalently *anomalous*) TQFT. We explain the sense in which this is an analogue of the classical spin representation. This is an instance of a phenomenon emphasized by Freed [Fre23]: Quantum theory is projective.

We also establish, under some assumptions, a general relationship between the language of projectivity/anomalies and the language of topological symmetries. We also identify a universal anomaly associated with any theory which is appropriately "simple".

Contents

1. Introduction 2
1.1. Anomalies 5
1.2. Chern-Simons theory 10
1.3. Anomalies in the Langlands program 11
1.4. Gapped systems 11
1.5. Sandwiches and anomalies 11
1.6. Gauging and anomalies 13
2. Projective TQFTs 14
2.1. Preliminaries: arrow categories 14
2.2. The projectivization of a target category 15
2.3. Fully-extended projective target categories 16
2.4. Projective theories 18
2.5. Extensions of the bordism category 19
3. Anomalies 19
3.1. Definitions 19
3.2. The anomaly associated to a projectivity class 21
3.3. Anomalies and sandwiches 22
3.4. A universal anomaly 23
4. Fusion categories 24
4.1. Preliminaries 24
4.2. Higher groupoids attached to a braided fusion category 26
4.3. Pointed braided fusion categories 27
4.4. Obstruction theory 30
4.5. Extensions of the finite orthogonal group 33

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4.6. Interlude: spinors 34
4.7. A detailed analogy with Spin 36
5. Projective 3-dimensional TQFTs 37
5.1. Dualizability and invertibility 37
5.2. WRT theories 38
5.3. The anomaly and the center 40
5.4. Projective action of the braided automorphisms of the center 40
5.5. Projective action of the orthogonal group 41
5.6. Projective 3d TQFTs 44
5.7. Functorial assignment of TQFTs to a vector space over a finite field 45
5.8. Module structures on 3d TQFTs 45
Appendix A. TQFT and category theory 46
A.1. The Cobordism Hypothesis 46
A.2. Relative theories 48
A.3. Invertible theories 49
Appendix B. Topological symmetry 49
B.1. TQFTs associated to π-finite spaces 49
B.2. Module structures 52
B.3. Reduction of topological symmetry 53
References 55

1. Introduction

Quantum field theory is fundamentally projective [Fre23, §3]. For instance, the correlation functions of a given QFT are unchanged by tensoring with an invertible theory. One example of this phenomenon, in quantum mechanics, is the well-known fact that the pure states form the *projectivization* of the Hilbert space of all states. From the perspective of symmetries, this projectivity appears when one asks if a certain group acts coherently on the Hilbert space: The (symplectic) group of symmetries of the phase space will only act projectively, i.e. only on the pure states.

This projectivity is captured by what is called an *anomaly*. Though anomalies often assume various mathematical guises, we will work with anomalies which are captured by invertible once-categorified theories. Anomalous theories are then theories defined relative to these invertible theories [Fre23].

We define the equivalent notion of a *projective* TQFT defined on $\mathbf{Bord}_d^X$, following suggestions of Freed [Fre23]. Namely, we formally define a "projectivization" construction, $\mathbb{P}$, of a linear target category (Definition 2.3). For example, $\mathbb{P}\mathbf{Vect}$ is an avatar of the category of projective spaces (see Remark 2.14) and serves as an appropriate target for non-extended projective TQFTs of any dimension. The definition of the projectivization (Definition 2.3) is general enough to apply to nearly any target category of interest, and manifestly classifies anomalous theories valued in that category. For example, by forming the projectivizations of the even higher Morita categories [JFS17], we obtain concrete target categories for projective TQFTs in any dimension.

We define the notion of a non-anomalous framed theory in any dimension having an anomaly as an X-theory in Definition 3.4. I.e. it is well-defined as a functor out of $\mathbf{Bord}_d^{\mathrm{fr}}$, but it is only a relative theory on $\mathbf{Bord}_d^X$. In particular, this includes the notion of a framing anomaly (Example 3.6), as well as the 't Hooft anomalies discussed in this paper. We discuss a target for projective 3-dimensional TQFTs in detail (Section 2.3.3). We also show that $\mathbb{P}$ commutes with iterated looping (Theorem 2.4) meaning, for example, that *projective* 3-dimensional TQFTs give rise to *projective* mapping class group representations of surfaces.

Under some assumptions (see Hypothesis Q), anomalous/projective theories can be rephrased via the language of topological symmetry [FMT22]. Let G be a finite group. If the anomaly theory, as a once-categorified TQFT on $\mathbf{Bord}_d^{BG}$, happens to be trivializable, then this is equivalent to having a "sandwich structure" [FMT22] (i.e. an 'action') of pure topological G-gauge theory of one dimension higher. This is still possible if the anomaly theory is nontrivial: Now the topological G-gauge theory is twisted by the cocycle underlying the anomaly theory (see Remark A.15). This statement identifies certain higher G-representations with certain higher modules over the higher group algebra of G.

For general π-finite X, we formalize this in Theorem 3.12 as an equivalence between four different avatars of projectivity. We also define a universal anomaly (Corollary 3.17) for an object of any target satisfying a certain notion of simplicity (Definition 3.14).

Let us restrict our attention to anomalies of 3-dimensional theories. The anomaly theories themselves will be valued in the (even higher [JFS17]) Morita 4-category[1] of braided tensor categories **BrTens**. We will restrict our attention to the subcategory consisting of rigid, finite, and semisimple tensor categories, i.e. fusion categories. One reason for this is that they are fully-dualizable [BJS21].

As usual, endomorphisms of the identity forms the associated 3-dimensional target: the Morita category of fusion categories, **Fus**. As is discussed in Section 2.2, the projectivization of **Fus**, written $\mathbb{P}\mathbf{Fus}$, is constructed in such a way that functors

$$\mathbf{Bord}_3^{(X,\zeta)} \to \mathbb{P}\mathbf{Fus} \tag{1.1}$$

(see Section A.1.1 for the notation $\mathbf{Bord}_d^{(X,\zeta)}$) correspond precisely to natural transformations between the unit and any invertible representation of the same bordism category on $\mathbf{BrFus}^\times$.

One justification for restricting to this dimension and setting, is that $\pi_0\mathbf{BrFus}^\times$ is highly nontrivial, and closely related to the Witt group of braided fusion categories [BJSS21]. This means there are more anomaly theories valued in **BrFus** than there are in, say, the category of vector spaces or the Morita category of algebras (or fusion categories), since these three categories have trivial groups of invertible objects.

An important example of a 3-dimensional TQFT is the Witten-Reshetikhin-Turaev (WRT) TQFT associated to a modular tensor category [Wit89, RT91, Tur94]. Given a nondegenerate braided fusion category $\mathcal{A}$, the avatar of this theory which we will consider is the TQFT $\mathbf{WRT}_{\mathcal{A}}\colon \mathbf{Bord}_3^{\mathrm{fr}} \to \mathbb{P}\mathbf{Fus}$, defined by sending the point to the fully-dualizable object given by $\mathcal{A}$ as a module over itself [Fre12, Haï24]. We prove that this theory always defines a projective G-theory, for any G acting fully-coherently by braided autoequivalences of $\mathcal{A}$ in Theorem 5.3.

WRT theories do not extend to the point as oriented (or even framed) TQFTs valued in **Fus** whenever $\mathcal{A}$ is in a nontrivial Witt class. This illustrates one feature of the projective target category $\mathbb{P}\mathbf{Fus}$: it allows for WRT theories (with a nontrivial framing anomaly) to be extended to the point.

Drinfeld centers of fusion categories are examples of braided fusion categories $\mathcal{A}$ which are in the trivial Witt class [BJSS21]. Every fusion category $\mathcal{C}$ is fully-dualizable [DSPS20], so it defines a fully-extended framed 3-dimensional TQFT by the Cobordism Hypothesis [Lur09]. This theory is sometimes called the Turaev-Viro (TV) theory associated with $\mathcal{C}$.

The TV theory for $\mathcal{C}$ is related to the WRT theory (in the above sense) associated to $\mathcal{Z}(\mathcal{C})$ [BK10, Bal10a, Bal10b], as follows: The anomaly theory of $\mathbf{WRT}_{\mathcal{Z}(\mathcal{C})}$ is trivializable: $\mathcal{A}$ is in the trivial Witt class, and a choice of such a $\mathcal{C}$ determines a trivialization of the anomaly. Composing

[1] We will generally use n-category to denote (∞, n)-category in the complete Segal space formalism, and otherwise we specify that we are considering a discrete n-category. We will sometimes still write (∞, n)-category to emphasize the ∞-categorical nature of the target categories. In particular, we work with the even higher Morita category of $\mathbb{E}_2$-algebras in (a suitable subcategory of) the symmetric monoidal 2-category of presentable categories. See Remark 4.1 for the relationship with the abelian category theory literature.

with the trivialization yields an absolute 3-dimensional theory, which sends the point to

$$\mathcal{C} \otimes_{\mathcal{Z}(\mathcal{C})} \mathcal{Z}(\mathcal{C}) \simeq \mathcal{C} \in \mathbf{Fus} ,$$

and therefore agrees with the TV theory for $\mathcal{C}$. This is a sense in which "TV for $\mathcal{C}$ agrees with WRT for $\mathcal{Z}(\mathcal{C})$".

The anomaly/projectivity of the TV theory only appears when we attempt to upgrade the theory to be defined on a bordism category with different background fields. An instance of this is given by the question of upgrading a theory to be equivariant for the action of a certain group. The anomaly is then encoded by the Postnikov tower of the higher automorphism groupoid of the fusion category.

The homotopy theoretic structure of the groupoid of automorphisms is studied in [ENO10]. We utilize the obstruction theory contained therein to produce certain 3-groups (i.e. twice-categorified groups) (Definitions 4.46, 4.48, and 4.52). These are central[2] extensions of finite orthogonal groups by higher scalars (e.g. $B^2\mathbb{C}^\times$ or $B^2\mu_p$). A detailed analogy with the classical spin representation is developed in Section 4.7. One can also think of this as being analogous to the Weil representation of the metaplectic group.

These 3-group extensions of the finite orthogonal group each act on the framed TQFT. Equivalently, the orthogonal group acts projectively, with projectivity given by the cocycle classifying the extension. In terms of anomalies, the original framed theory defines an anomalous/projective theory on $\mathbf{Bord}_3^{BG}$, where G is any group mapping to the finite orthogonal group (Theorem 5.17 and Corollary 5.20). When the fusion category in question consists of vector spaces graded by $\mathbb{F}_p^n$, the obstruction / anomaly is trivializable. Therefore we obtain the existence of an $\mathrm{O}(n,n;\mathbb{F}_p)$-equivariant *linear* TQFT associated to $\mathbb{F}_p^n$ (Corollary 5.28). Under the analogy with the Weil representation, the fact that this class is trivializable is analogous to the splitting of the Weil representation over a finite field [GH09, GH11].

On the other hand, given any finite group G and 4-cocycle we show in Corollary 5.27 that this cocycle is the pullback of the class k_3 associated to some nondegenerate braided fusion category.

This paper is organized as follows. In the remainder of the introduction, we give a more detailed summary of our results and give some indication as to how this work is related to the literature.

In Section 2 we implement suggestions of Freed [Fre23] to provide a formal definition of a projective TQFT. We define the projectivization functor in Section 2.2. We apply it to various (even higher) Morita categories in Section 2.3, where we also explain some concrete examples of projective targets. The definition of a projective TQFT is in Section 2.4, and the relationship with extensions of bordism categories is discussed in Section 2.5.

In Section 3 we discuss anomalies in general. In Section 3.2 we explicitly describe how a projectivity cocycle classifies a once-categorified invertible TQFT, following [Fre23]. In Sections 3.3 and 3.4 we provide the connection with the sandwich picture in Theorem 3.12 and the "universal anomaly" in Corollary 3.17.

In Sections 4.1, 4.2, 4.3, and 4.4 we collect the necessary preliminaries regarding (braided) fusion categories, mostly from [ENO10]. In Section 4.5 we introduce the 3-groups which are processed into anomaly theories, and in Sections 4.6 and 4.7 we explain a detailed analogy between these 3-groups and the groups Pin and Spin.

In Sections 5.1 and 5.3 we review some facts concerning fully-extended 3-dimensional TQFTs and Drinfeld centers, before moving onto anomalies. A projective action of the braided automorphism 2-group of the center (resp. orthogonal group of the underlying metric group) is considered in Section 5.4 (resp. Section 5.5), and their avatars as *projective* theories are discussed in Section 5.6. The anomalies/projectivity are translated into the language of topological symmetries in Section 5.8. The functorial assignment of a linear $\mathrm{O}(n,n;\mathbb{F}_p)$-equivariant 3-dimensional TQFT to a vector space over $\mathbb{F}_p$ is considered in Section 5.7.

[2] See Remark 4.41.

In Appendix A we recall general facts about TQFTs, mostly from [Lur09]. In particular, we discuss the particular version(s) of the Cobordism Hypothesis we will use in this paper in Section A.1. Relative TQFTs and invertible TQFTs are considered in Sections A.2 and A.3.

Appendix B is dedicated to a discussion of topological symmetry and TQFTs associated to π-finite groupoids. We review some definitions and facts from [FMT22], and define a notion of a "reduction" of topological symmetry in Section B.3.

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1.1. **Anomalies.** Notation which is not completely standard is introduced in the body of the paper: $\mathbf{Bord}_d^{(X,\zeta)}$ and $\mathcal{T}^{\sim}$ in Section A.1, the arrow category $\mathcal{T}^{\downarrow}$ in Section 2, $\mathbb{P}$ in Definition 2.3, quiches and Hypothesis Q in Appendix B, 3Lip, 3Pin, and 3Spin in Section 4.5, and polarized metric groups in Section 4.3.6.

Anomalies have been studied for many years, under many different guises. Early appearances include [Ste49, Adl69, BJ69, tH80], and more contemporary resources, which directly inspired this paper, include [Fre23, Fre14a, FHLT10]. See [Fre23] and the references contained therein for more thorough resources. We take the point of view that an anomaly is a feature, rather than a bug [tH80, Fre23]. Anomalies can, and often are, discussed for more general classes of QFTs, but we will restrict our attention to topological anomalies of TQFTs. The main proposal of this work is that the Postnikov tower of the higher automorphism groupoid is a source of anomalies/projectivity.

This notion of an anomalous theory is closely tied to the notion of a *projective* TQFT, as outlined in [Fre23]. We introduce projective target categories for such theories in Section 2.2. Given any symmetric monoidal (∞, d)-category (not necessarily linear), in Definition 2.3 we give a formal definition for a *projectivization* of any such target. A d-dimensional projective TQFT is then a functor from the bordism category of interest to the projectivization of the target category of interest. In Proposition 3.5 we record the nearly tautological statement that this notion of a projective theory is equivalent to an anomalous theory.

Given any sufficiently nice (see Remark 2.12 for the precise condition) symmetric monoidal (∞, k)-category $\mathcal{S}$, the even higher Morita category [JFS17] $\mathbf{Alg}_n\mathcal{S}$ is a symmetric monoidal $(\infty, k+n)$-category. The projectivization $\mathbb{P}\mathbf{Alg}_n\mathcal{S}$ is a candidate target for projective $(k+n)$-dimensional TQFTs. This gives a well-defined notion of a (fully-extended) projective TQFT in any dimension. An explicit example of interest is the projectivization of **Vect** (considered in Section 2.3.2), which can serve as a target for non-extended TQFTs of any dimension.

Let $\mathcal{T}$ denote the $(d+1)$-dimensional target (i.e. the target for bulk theories). Write $\Omega\mathcal{T} = \mathrm{End}_{\mathcal{T}}(1)$, and Ω^j for the j-fold endomorphisms of the unit. We show in Theorem 2.4 that $\mathbb{P}$ depends naturally on $\mathcal{T}$, and that $\mathbb{P}$ commutes with Ω^j. In particular, if the linear category $\mathcal{T}$ satisfies $\Omega^d\mathcal{T} \simeq \mathbf{Vect}$, and every invertible object of $\Omega^{d-1}\mathcal{T}$ is trivializable (see Warning 1), then we have that:

$$\Omega^{d-1}\mathbb{P}\Omega\mathcal{T} \simeq \mathbb{P}\Omega^d\mathcal{T} \simeq \mathbb{P}\mathbf{Vect} \ .$$

Since $\mathbb{P}\mathbf{Vect}$ is an avatar of the category of projective vector spaces (see Remark 2.14), one upshot of this statement is that the mapping class group representations that come from a *projective* TQFT are indeed *projective* mapping class group representations.

Remark 1.2 (Framing anomalies). The results in this paper concerning three-dimensional theories are restricted to the framed setting, in order to avoid the discussion of an SO-fixed point structure on (braided) fusion categories. If a given theory cannot be upgraded from a framed theory to an oriented theory in this manner, then it is said to have a nontrivial *framing anomaly*. (See Example 3.6 for a formal definition in terms of invertible once-categorified theories.) The anomalies in this paper which correspond to projective actions of (higher) groups are not entirely of the same sort as a framing anomaly: The tangential structure on our source $\mathbf{Bord}_d^X$ (borrowing notation from [Lur09]) is the trivial map $X \to B\,\mathrm{O}\,(d)$.

This is not to say that the discussion in this paper is entirely orthogonal to the discussion of framing anomalies: The four-dimensional Crane-Yetter theory (see Section 5.1) plays an important role in studying the framing anomaly of the 3-dimensional Reshetikhin-Turaev theory [RT90, RT91, Wit89, Wal91, FHLT10].

1.1.1. *Fully-extended anomalous WRT theory.* Unless otherwise specified, all categories are $\mathbf{k}$-linear, where $\mathbf{k}$ is algebraically closed of characteristic zero. The anomaly theories of the anomalous 3-dimensional theories are valued in the Morita 4-category of *braided* fusion categories,[3] written $\mathbf{BrFus}$. By constructing the projectivization of $\Omega\mathbf{BrFus} \simeq \mathbf{Fus}$ (where $\mathbf{Fus}$ is the Morita 3-category of fusion categories) we are able to introduce a concrete definition of a *projective* 3-*dimensional TQFT* in Section 5.6. This is of particular interest because the low energy *topological* theory modeling a gapped system is in fact a *projective*, i.e. anomalous, TQFT [Fre23, Interlude (P.12)].

The objects of $\mathbb{P}\mathbf{Fus}$ are 1-morphisms in $\mathbf{BrFus}$ from an invertible object to the unit. Spelling this out, an object consists of some invertible object $\mathcal{A}$ of $\mathbf{BrFus}$, along with an $\mathbb{E}_1$-algebra object of the monoidal 2-category of $\mathcal{A}$-modules. For instance, the regular module $\mathcal{A}_{\mathcal{A}}$ is such a 1-morphism. As an object of the arrow category $\mathbf{BrTens}^{\downarrow}$, any such regular module defines a fully-dualizable 1-morphism [Haï24].

The upshot of this discussion, is that $\mathcal{A}$ defines a projective TQFT:

$$\mathbf{WRT}_{\mathcal{A}} \colon \mathbf{Bord}_3^{\mathrm{fr}} \to \mathbb{P}\mathbf{Fus}\ .$$

The object $\mathcal{A} \in \mathbf{BrFus}$ is invertible on account of being nondegenerate [BJSS21]. Therefore the TQFT $\mathbf{CY}_{\mathcal{A}} \colon \mathbf{Bord}_4^{\mathrm{fr}} \to \mathbf{BrFus}$ sending the point to $\mathcal{A}$ is also invertible. It is a fully-extended framed version of the Crane-Yetter TQFT attached to $\mathcal{A}$, and it is the anomaly theory of $\mathbf{WRT}_{\mathcal{A}}$.

We show that $\mathbf{WRT}_{\mathcal{A}}$ can always be given a projective action of the 2-group of braided autoequivalences of $\mathcal{A}$. The following appears as Theorem 5.3 in the text.

Theorem. *Let $\mathcal{A}$ be a nondegenerate braided fusion category, and consider a monoidal functor $\Phi \colon G \to \mathrm{Aut}_{\mathbf{EqBr}}(\mathcal{A})$. There is an anomalous G-TQFT*

$$\mathbf{WRT}_{\mathcal{A}}^{G} \colon \mathbf{Bord}_3^{BG} \to \mathbb{P}\mathbf{Fus}$$

which agrees with $\mathbf{WRT}_{\mathcal{A}}$ upon restriction to trivial G-bundles. The anomaly theory of $\mathbf{WRT}_{\mathcal{A}}^{G}$, as a functor from $\mathbf{Bord}_4^{BG}$ to $\mathbf{BrFus}$, agrees with $\mathbf{CY}_{\mathcal{A}}$ upon restriction to trivial G-bundles.

In words, this theorem says that we can upgrade WRT and Crane-Yetter to be G-theories in such a way that WRT still lives relative to Crane-Yetter.

Remark 1.3. As in [BJS21, BJSS21], we will prefer to work in the more general ∞-categories consisting of objects which are Ind-completions of what are usually called fusion categories in the abelian category literature. As explained in Remark 4.1, various existing results allow us to

[3] See Section 4.1 for detailed definitions.

translate facts back and forth between these two settings [DSPS20, BJS21, BJSS21]. For example, we make frequent use of facts from [ENO10]. Our perspective is that this is a feature not a bug: We can continue to use the arguments from the finite abelian category theory literature, however the target categories considered in this paper contain more general objects. For example, the iterated looping to a 1-category is the Ind-completion of the category of finite-dimensional vector spaces, which in-particular contains objects which represent infinite-dimensional vector spaces.

1.1.2. *An anomaly of a fully-extended framed TV theory.* Let $\mathcal{C}$ be a fusion category (see Remark 4.1 for the details). The Morita 3-category of fusion categories, **Fus**, has duals [DSPS20], so $\mathcal{C}$ classifies a (framed) 3-dimensional TQFT F by the Cobordism Hypothesis [Lur09].

The theory F is well-defined as a framed theory. However, F will turn out to be "anomalous" or "projective" as a theory which is equivariant with respect to a certain natural group, or in some cases 2-group, of symmetries. In physical terms, we are identifying an obstruction to gauging the symmetry.

The Drinfeld center $\mathcal{Z}(\mathcal{C})$ is an object of the 4-category **BrFus**, which also has duals [BJS21], so this classifies a 4-dimensional TQFT ζ. The Drinfeld center $\mathcal{Z}(\mathcal{C})$ tautologically acts on $\mathcal{C}$, meaning the theory F can be upgraded to a relative theory $F_\zeta \colon \zeta \to 1$. The Drinfeld center turns out to be invertible [BJSS21], and in fact $\mathcal{Z}(\mathcal{C})$ is trivializable in **BrFus**: The module $\mathcal{C}$ in $\mathrm{Hom}_{\mathbf{BrFus}}(\mathcal{Z}, 1)$ is an equivalence. I.e. $F_\zeta \colon \zeta \to 1$ is an equivalence of theories. The anomalous symmetries of the original theory F appear when we consider the symmetries of ζ, or equivalently $\mathcal{Z}(\mathcal{C})$.

In [ENO10], the homotopy type of the 3-type $B\,\mathrm{Aut}_{\mathbf{Fus}}(\mathcal{C})$ is studied. There is a natural map:

$$\mathrm{Aut}_{\mathbf{Fus}}(\mathcal{C}) \to \mathrm{Aut}_{\mathbf{EqBr}}(\mathcal{Z}(\mathcal{C})) \ ,$$

where $\mathrm{Aut}_{\mathbf{EqBr}}$ denotes the 2-group of braided autoequivalences. As it turns out, this map is an equivalence when restricted to the truncation of $\mathrm{Aut}_{\mathbf{Fus}}(\mathcal{C})$ to a 2-group. The top nontrivial homotopy group of $B\,\mathrm{Aut}_{\mathbf{Fus}}(\mathcal{C})$ is $\pi_3 = \mathbf{k}^\times$, so the upshot of this discussion is that the higher automorphism groupoid defines a bundle:

$$\begin{array}{ccccc} B^3\mathbf{k}^\times & \longrightarrow & B\,\mathrm{Aut}_{\mathbf{Fus}}(\mathcal{C}) & \longrightarrow & * \\ \downarrow & & \downarrow & & \downarrow \\ * & \longrightarrow & B\,\mathrm{Aut}_{\mathbf{EqBr}}(\mathcal{Z}(\mathcal{C})) & \xrightarrow{k_3} & B^4\mathbf{k}^\times \end{array} \tag{1.4}$$

The map k_3 defines a TQFT:

$$\alpha_{k_3} \colon \mathbf{Bord}_4^{B\,\mathrm{Aut}_{\mathbf{EqBr}}(\mathcal{Z}(\mathcal{C}))} \to \mathbf{BrFus} \ .$$

The original theory F canonically defines a relative theory:

$$F_{k_3} \colon \alpha_{k_3} \to 1 \ ,$$

where now 1 denotes the trivial theory with source $\mathbf{Bord}_4^{B\,\mathrm{Aut}_{\mathbf{EqBr}}(\mathcal{Z}(\mathcal{C}))}$. (See Proposition 3.8.)

I.e. we have seen that the ordinary framed theory F is well-defined as an anomalous $\mathrm{Aut}_{\mathbf{EqBr}}(\mathcal{Z}(\mathcal{C}))$-equivariant theory. The following is stated as Theorem 5.10 in the body of this paper.

Theorem. *The framed TQFT F has an $X = B\,\mathrm{Aut}_{\mathbf{EqBr}}(\mathcal{Z}(\mathcal{C}))$-anomaly as in Definition 3.4. In particular, the fusion category $\mathcal{C}$ itself defines an anomalous theory:*

$$F_{k_3} \colon \alpha_{k_3} \to 1_{B\,\mathrm{Aut}_{\mathbf{EqBr}}(\mathcal{Z}(\mathcal{C}))} \ .$$

*Furthermore, if $\mathcal{Z}(\mathcal{C})$ is pointed, and the cohomology class classifying the braiding of $\mathcal{Z}(\mathcal{C})$ (as in Remark 5.2) is nontrivial (e.g. whenever $\mathcal{C} \not\simeq$ **Vect**), then the class k_3 classifying the anomaly α_{k_3} is nontrivial.*

If $\mathcal{Z}(\mathcal{C})$ is pointed (and the group of isomorphism classes of simple objects A has odd order) then the 2-group $\mathrm{Aut}_{\mathbf{EqBr}}(\mathcal{Z}(\mathcal{C}))$ splits, and in fact there is a canonical splitting. I.e. we can (canonically) write the classifying space as a semidirect product of Eilenberg-Mac Lane spaces

$B\pi_1$ and $B^2\pi_2$. The fundamental group is $\pi_1 = \mathrm{O}(A,q)$, where (A,q) is the pre-metric group classifying the pointed braided fusion category $\mathcal{Z}(\mathcal{C})$ (see Section 4.3); we will often write $\mathrm{O}(A)$ for $\mathrm{O}(A,q)$. The upshot of this is that we can pull the class k_3 back to $B\,\mathrm{O}(A)$, along this canonical splitting, to obtain a class:

$$O_4(A) \in H^4\left(B\,\mathrm{O}(A), \mathbf{k}^\times\right) \ .$$

This defines an anomaly theory:

$$\alpha_{O_4} \colon \mathbf{Bord}_4^{B\,\mathrm{O}(A)} \to \mathbf{BrFus} \ ,$$

and F defines a relative theory:

$$F_{O_4} \colon \alpha_{O_4} \to 1 \ .$$

The homotopy fiber of $O_4(A)$ is a $B^3\mathbf{k}^\times$-bundle over $B\,\mathrm{O}(A)$. This is the classifying space of a 3-group we call $3\mathrm{Lip}(A)$ (Definition 4.46). This is an analogue of the Lipschitz (a.k.a. Clifford) group, reviewed in Section 4.6. As in the classical story, $3\mathrm{Lip}(A)$ can be cut down to an extension by a finite group, rather than all scalars. We call this 3Pin (Definition 4.48), and it is analogous to the group Pin. We can restrict to $\mathrm{SO}(A) \hookrightarrow \mathrm{O}(A)$ to obtain 3Spin (Definition 4.52), an analogue of Spin. This is a part of an analogy in Table 1, which is fleshed out in detail in Section 4.7.

TABLE 1. A detailed analogy between the Clifford algebra and spin representation associated to (a Lagrangian in a) quadratic vector space, and the braided fusion category and fusion module category associated to (a Lagrangian in a) finite metric group. See Section 4.6 for the former, and Section 4.5 for the latter. This analogy is explained in detail in Section 4.7.

1-dimensional	3-dimensional
(V,q)	(A,q)
$\mathrm{SO}(V,q) \subset \mathrm{O}(V,q)$	$\mathrm{SO}(A,q) \subset \mathrm{O}(A,q)$
$\mathbf{k}^\times$	$B^2\mathbf{k}^\times$
$\mathrm{Cliff}(V)$	$\mathcal{A} = (\mathbf{Vect}[A], *, \beta_q)$
$\{x,y\} = b_q(x,y)$	$\beta_{b,a} \circ \beta_{a,b} = \langle a,b\rangle_q \,\mathrm{id}$
$V \rtimes \mathrm{O}(V,q)$	$\mathrm{Aut}_{\mathbf{EqBr}}(\mathcal{A})$
$\Gamma = \mathrm{Lip}$	$3\mathrm{Lip}(A,q)$
$\mathrm{Pin}(V,q)$	$3\mathrm{Pin}(A,q)$
$\mathrm{Spin}(V,q)$	$3\mathrm{Spin}(A,q)$
$\{\pm 1\} \hookrightarrow \mathbf{k}^\times$	$B^2\mu_{l^4} \to B^2\mathbf{k}^\times$
See Remark 4.58	$B^2\mathcal{A}^\times$
See Remark 4.58	$\mathrm{Pic}(\mathcal{A})$
$V \simeq L \oplus L^*$	$A \simeq L \oplus L^*$
$\wedge^\bullet L^*$	$\mathcal{C} = (\mathbf{Vect}[L^*], *)$
$\mathrm{End}(\wedge^\bullet L^*) \simeq \mathrm{Cliff}$	$\mathrm{Aut}_{\mathbf{Fus}}(\mathcal{C}) \simeq \mathrm{Pic}(\mathcal{A})$

The following results are stated as Theorem 5.17 and Corollary 5.20 in the body of this paper.

Theorem. *Consider a pointed Drinfeld center $\mathcal{Z}(\mathcal{C})$ classified by a polarized metric group (A, q, L) with $|A|$ odd. The nonanomalous framed theory*

$$F\colon \mathbf{Bord}_3^{fr} \to \mathbf{Fus}$$

sending the point to $\mathcal{C}$ has an $\mathrm{O}(A,q)$-anomaly in the sense of Definition 3.4. I.e. there is an anomaly theory

$$\alpha_{c(3\mathrm{Pin})}\colon \mathbf{Bord}_3^{B\,\mathrm{O}(A,q)} \to \mathbf{BrFus}\ ,$$

and $\mathcal{C}$ canonically defines an anomalous $\mathrm{O}(A,q)$-TQFT:

$$F_{c(3\mathrm{Pin})}\colon \alpha_{c(3\mathrm{Pin})} \to 1_{B\,\mathrm{O}(A,q)}\ .$$

Corollary. *Restricting the $\mathrm{O}(A,q)$-anomaly of Theorem 5.17 to $\mathrm{SO}(A,q)$ we obtain an $\mathrm{SO}(A,q)$-anomaly of F*

$$\alpha_{c(3\mathrm{Spin})}\colon \mathbf{Bord}_3^{B\,\mathrm{SO}(A,q)} \to \mathbf{BrFus}\ ,$$

and an anomalous $\mathrm{SO}(A,q)$-equivariant theory

$$F_{c(3\mathrm{Spin})}\colon \alpha_{c(3\mathrm{Spin})} \to 1_{B\,\mathrm{SO}(A,q)}\ .$$

If one must trivialize the anomaly[4], and it is not trivializable on all of π_1, it can be pulled back along a map $f\colon G \to \pi_1$. This results in an anomaly:

$$\alpha\colon \mathbf{Bord}_4^{BG} \to \mathbf{BrFus}\ ,$$

and F still defines a relative theory $\alpha \to 1$. A *trivialization* is an equivalence: $1 \xrightarrow{\sim} \alpha$, and the *trivialized theory* is the composition:

$$1 \xrightarrow{\sim} \alpha \to 1\ .$$

Note that this is an endomorphism of the trivial theory defined on $\mathbf{Bord}_4^{BG}$, and therefore equivalent to a theory

$$F_G\colon \mathbf{Bord}_3^{BG} \to \mathbf{Fus}\ .$$

One should think that we started with a theory F (or fusion category $\mathcal{C}$) and obtained a G-equivariant theory F_G (or fusion category $\mathcal{C}$ along with fully coherent[5] action of G on $\mathcal{C}$ via bimodules).

Remark 1.5. By construction, the direct sum of these bimodules for each $g \in G$ is precisely the G-extension associated to the same data in [ENO10, Theorem 1.3].

Remark 1.6. Sometimes the entire cohomology group containing the obstruction vanishes, meaning the anomaly is trivializable. One example of this is $\mathcal{C} = \mathbf{Vect}[L]$, for L a vector space over $\mathbb{F}_p$. In this case, the (cohomology group and therefore the) obstruction is shown to vanish in [EG18]. In the language of this paper, this means that there is an $\mathrm{O}(L \oplus L^*, \mathrm{ev})$-equivariant theory sending the point to $\mathcal{C}$. This is stated as Corollary 5.28 in the text.

[4] An anomaly is really "part" of the theory, and only acts as an obstruction when you are "quantizing", see [Fre23, §4].

[5] Fully coherent means that the assignment of a bimodule to each group element is not only defined up to isomorphism: it is a functor from the discrete 3-category with objects G to the full 3-group of $\mathcal{C}$-bimodules. See Section 5.5.2 for more.

1.1.3. *Characteristic classes.* Anomalies are most interesting when they cannot be trivialized as in Section 5.5.2. This happens, for example when we consider the full symplectic group acting on the quantum harmonic oscillator: The Hilbert space $L^2(\mathbb{R})$ is a nontrivially projective representation of $\mathrm{Sp}_2(\mathbb{R})$, and the projectivity classifies the universal cover $\mathrm{Mp}_2(\mathbb{R}) \to \mathrm{Sp}_2(\mathbb{R})$. The corresponding cocycle classifies (a multiple of) $w_2 \in H^2(B\,\mathrm{Sp}_2(\mathbb{R}))$.

Similarly, finite orthogonal groups have characteristic classes [FP78] and one might wonder if these characteristic classes match the anomaly theories discussed in this paper.

More generally, given a 4-cocycle $\pi\colon BG \to B^4\mathbf{k}^\times$ on a group, one might wonder what conditions guarantee the existence of an anomalous 3-dimensional TQFT (thought of as a nondegenerate braided fusion category $\mathcal{B}$) such that the anomaly is described by π. Corollary 5.27 provides a partial answer for finite G.

1.1.4. *Relationship with the literature.* The anomalies studied in this paper are 't Hooft anomalies for discrete internal symmetries of Dijkgraaf-Witten theories. These were first studied (in various dimensions) in [KT14a, KT14b], and (in all dimensions) in the once-extended functorial language in [Mül20, MS20].

In the 3-dimensional and pointed/quasi-trivial case, the obstruction classes in [KT14a, Mül20, MS20] are shown (in [ENO10, §11.8]) to be equivalent to the obstruction classes in [ENO10].

The higher groupoids studied in [ENO10] were studied as symmetries of 3-dimensional theories in [FPSV15, FS16]. The obstruction theory of [ENO10] is used in Section 4.4 to define the 3-groups which characterize the anomaly theories. The same obstruction theory used in this paper from [ENO10] has been used extensively in the literature to study MTCs in the unitary setting [CGPW16, DGP$^+$21, DGP$^+$23].

A partial description of the Brauer-Picard 3-group of the Asaeda-Haagerup fusion categories was given in [GJS15]. The missing information was precisely this k-invariant which classifies the anomaly studied herein, which was shown to vanish for these examples in [GIS18].

The finiteness results of [DSPS20, BJS21, BJSS21, Haï24] are referenced in Section 5.1 to construct fully-extended, and sometimes invertible, TQFTs which agree with the theories associated with certain π-finite spaces (in the sense of [FMT22, §A.2]).

The original Crane-Yetter and Reshetikhin-Turaev invariants/TQFTs [RT90, RT91, TV92, CY93, Del01] are believed to agree with (oriented upgrades) of the theories discussed in Section 5.2 and Section 5.3. In particular, Theorem 5.3 is consistent with the idea that Crane-Yetter theory encodes an anomaly of the Reshetikhin-Turaev theory associated to the MTC [Wal91, Haï25].

The notion of a projective 3-dimensional TQFT is presumably closely related to the notion of a "modular functor" [Seg89, MS89, Tur94, Til98, BK01, BW23], which is roughly a system of projective representations of mapping class groups. Indeed, any projective 3-dimensional TQFT gives rise to a projective mapping class group representation for any surface. A formal comparison would be quite interesting, but we do not pursue this here.

1.2. **Chern-Simons theory.** Let G be a compact Lie group, and $\lambda \in H^4(BG, \mathbb{Z})$. Consider the Reshetikhin-Turaev TQFT attached to G at level λ [RT90, RT91, BM05, Sti08, FT21]. Recall this sends the circle, with the framing which extends over the disk, to the (semisimplification of the) category of representations of the quantum group at fixed root of unity determined by λ.

When $G = T$ is a torus, write $\Pi = \mathrm{Hom}(\mathrm{U}(1), T)$ for the associated lattice. Then λ determines a nondegenerate symmetric bihomomorphism

$$\langle\,,\,\rangle : \Pi \times \Pi \to \mathbb{Z}\ .$$

This induces a homomorphism $T \to T^*$ with kernel given by a finite group A. This finite group inherits the quadratic form associated to $\langle\,,\,\rangle$, and therefore becomes a nondegenerate metric group (A, q) as in Section 4.3.1. See [FHLT10, §9.3] for the role played by this finite group in identifying the framing anomaly of Chern-Simons.

The main upshot of this, for us, is that the braided fusion category $\mathcal{B} = \mathbf{Vect}[A]$ with convolution and braiding from q (see Section 4.3.1) generates the same 1-2-3 theory as the Reshetikhin-Turaev 1-2-3 theory attached to the category of representations of the quantum group for T at level λ [FHLT10].

The problem of extending Chern-Simons theory to the point has received considerable attention [FHLT10, Hen17, FT21]. One way to extend this Reshetikhin-Turaev 1-2-3 theory to the point is by asking for a fusion category $\mathcal{C}$ such that $\mathcal{B} \cong \mathcal{Z}(\mathcal{C})$ as braided categories. Given that one such fusion category $\mathcal{C}$ exists, one might also wonder how uniquely it is determined. It is shown, in [ENO11], that two fusion categories are Morita equivalent if and only if the Drinfeld centers are equivalent as braided categories. This is strengthened in [ENO10] to an equivalence of 2-groups:

$$\tau_{\leq 1} \operatorname{Aut}_{\mathbf{Fus}}(\mathcal{C}) \simeq \operatorname{Aut}_{\mathbf{EqBr}}(\mathcal{Z}(\mathcal{C})) \ .$$

I.e. the isomorphism class of $\mathcal{C}$ in **Fus** is almost determined by the isomorphism class of $\mathcal{Z}(\mathcal{C})$ in **BrFus**: There is a lack of coherence at the top level, captured by an obstruction class originally studied in [ENO10] and surveyed in Section 4.4.3. This same lack of coherence is captured by the anomaly theory studied in Section 5.5.

1.3. **Anomalies in the Langlands program.** An analogous anomaly to the one studied here appears in the Langlands program [AV23, BZSV24]. Indeed, this was the main source of motivation for the project. More specifically, upgrading *Rozansky-Witten theory* [RW97, RW10, KRS09, KR10] to a boundary/relative theory for certain 4-dimensional gauge theories would produce a version of the relevant relative theory on the B-side (i.e. spectral side). The author hopes to return to these questions in the future.

1.4. **Gapped systems.** It is well-known that gapped phases of matter are described at low energy/long range by (unitary) TQFTs [Fre14b, FH21, FT21]. See also [RW18, KZ22]. In particular, it is pointed out in [Fre23, Interlude (P.12)], that the low energy linear theory describing the system is in fact *not* topological (it has a metric dependence [Wit89]), however its projectivization is indeed topological. In particular, this means *projective* 3-dimensional TQFTs model gapped systems in $(2+1)$-dimensions. We discuss such theories in Section 5.6.

't Hooft anomalies for discrete internal symmetries are particularly important for the study of SPT phases [Kap14, KT14a, KT14b, Mül20, MS20].

A related subject is the study of anyon systems. One way to model anyon systems, spelled out e.g. in [RW18], is using unitary modular categories (UMC). The question of whether or not the UMC is a Drinfeld center, which plays a big role in this paper, is also important in the study of anyon systems. For example, [RW18, Conjecture 4.2] states that a UMC being realized as a topological phase of matter is equivalent to being a Drinfeld center of a (unitary) fusion category.

When the fusion category $\mathcal{C}$ is the category of vector spaces graded by a finite group L, possibly twisted by a cocycle, the corresponding 3-dimensional theory is Dijkgraaf-Witten theory [DW90]. Kitaev introduced a Hamiltonian realization [Kit03], and for $L = \mathbb{Z}/2$ this is the famous toric code.

The structure of the 3-type $B \operatorname{Aut}_{\mathbf{Fus}}(\mathcal{C}) \simeq B \operatorname{Pic}(\mathcal{Z}(\mathcal{C}))$, as identified in [ENO10], is the source of the projectivity/anomalies identified in Section 5. The homotopy type of this groupoid is of much interest in the literature. For example, let $\mathcal{B}$ be an arbitrary unitary modular category. It is conjectured in [AWH22] that the homotopy type of $B \operatorname{Pic}(\mathcal{B})$ is related to a space of gapped Hamiltonians which give rise to the topological order described by $\mathcal{B}$.

1.5. **Sandwiches and anomalies.** The discussion in Section 1.1.2 fits into a more general picture, which relates to the language of topological symmetries [FMT22]. An upshot of this discussion, in the context of Section 1.1.2, will be that the anomalous theory in Theorem 5.17 (resp. Corollary 5.20) will equivalently define a $\sigma_{B\,\mathrm{O}(A,q)}$-module structure (in the sense of [FMT22], see Appendix B) twisted by the class which classifies the 3-group 3Pin (resp. 3Spin) over $\mathrm{O}(A,q)$.

Let $d \in \mathbb{Z}^{\geq 0}$, and let $\mathcal{T}$ be an $(\infty, d+1)$-category with duals. Given any TQFT

$$F \colon \mathbf{Bord}_d \to \Omega\mathcal{T}$$

we can consider the higher automorphism group of the assignment to the point: $\mathrm{Aut}_{\Omega\mathcal{T}}(F(*))$. Note that this is a d-group.

Assume that $B\,\mathrm{Aut}_{\Omega\mathcal{T}}(F(*))$ fibers over some space X with fiber $B^d\mathbf{k}^\times$, classified by some map c:

$$\begin{array}{ccccc} B^d\mathbf{k}^\times & \longrightarrow & B\,\mathrm{Aut}_{\Omega\mathcal{T}}(F(*)) & \longrightarrow & * \\ \downarrow & & \downarrow & & \downarrow \\ * & \longrightarrow & X & \xrightarrow{c} & B^{d+1}\mathbf{k}^\times \end{array}$$

This assumption is the stand-in for (1.4). For ease of exposition, write:

$$\widetilde{X} = B\,\mathrm{Aut}_{\Omega\mathcal{T}}(F(*)) \ .$$

Tautologically we have a theory:

$$\widetilde{F} \colon \mathbf{Bord}_d^{\widetilde{X}} \to \Omega\mathcal{T} \ ,$$

classified by the inclusion of $\widetilde{X}$ into $\Omega\mathcal{T}$. Now we ask if the theory $\widetilde{F}$ factors through/descends to a theory on $\mathbf{Bord}_d^X$:

$$\begin{array}{ccc} \mathbf{Bord}_d^{\widetilde{X}} & \longrightarrow & \Omega\mathcal{T} \\ \downarrow & \nearrow & \\ \mathbf{Bord}_d^X & & \end{array} \tag{1.7}$$

We will now rephrase this question in terms of twisted topological symmetry as in [FMT22]. In order to do this, we will assume Hypothesis Q, which asserts that there is a suitable association of TQFTs to π-finite groupoids.

We will now assume that X is π-finite. Recall we are assuming $\Omega^{d+1}\mathcal{T} \simeq \mathbf{k}$. Then we can consider the composition of the cocycle c with the inclusion of $B^{d+1}\mathbf{k}^\times$ into $\mathcal{T}$:

$$X \xrightarrow{c} B^{d+1}\mathbf{k}^\times \to \mathcal{T} \ ,$$

which classifies an invertible theory

$$\mathbf{Bord}_{d+1}^X \to \mathcal{T} \ . \tag{1.8}$$

If we truncate the invertible theory (1.8), we obtain a once-categorified invertible theory:

$$\alpha \colon \mathbf{Bord}_{d+1}^X \to \mathcal{T} \ .$$

As explained in [FMT22, §A.2], we regard the map from any given bordism to X as a fluctuating field, which can be integrated over to obtain a new $(d+1)$-dimensional theory $\sigma_{X,c}^{d+1}$.

Now the upshot is that a module structure (in the sense of [FMT22], see Appendix B) on the original theory F over $\sigma_{X,c}^{d+1}$ is equivalent to a projective TQFT with projectivity α.

The following, stated as Theorem 3.12 in the body of this paper, summarizes this discussion.

Theorem. *An X-anomaly $F_c \colon \alpha_c \to 1_X$ of F (as in Definition 3.4) naturally defines:*

- *(i) A projective X-theory $\overline{F} \colon \mathbf{Bord}_d^X \to \mathbb{P}\Omega\mathcal{T}$ with projectivity α_c and underlying theory F (as in Definition 2.17).*
- *(ii) $\widetilde{F} \colon \mathbf{Bord}_d^{\widetilde{X}} \to \Omega\mathcal{T}$ with underlying framed theory, written F.*
- *(iii) Assuming Hypothesis Q, a $(\sigma_{X,c}, \rho_{X,c})$-module structure on F (Section B.2), i.e. a realization of the quiche $(\sigma_{X,c}, \rho_{X,c})$ as symmetries of F, in the sense of* [FMT22].

Example 1.9. We have discussed this formalism in general, but it is most interesting when the space X has a novel interpretation. E.g. in Sections 5.4 and 5.5 we use the canonical identification of the truncation to a 2-type with the braided autoequivalences of the center [ENO10] to interpret this as an anomaly of F as a theory which is equivariant with respect to the braided autoequivalences of the center.

Slightly more generally, consider any nondegenerate braided fusion category $\mathcal{A}$. There is a fibration:

$$\begin{array}{ccc} B^3\mathbf{k}^\times & \longrightarrow & B(A\text{-}\mathbf{mod})^\times \\ \downarrow & & \downarrow{\scriptstyle \tau_{\leq 2}} \\ * & \longrightarrow & B\,\mathrm{Aut}_{\mathbf{EqBr}}(\mathcal{A}) \end{array}$$

by [ENO10, Theorem 5.2], which is classified by some map $c\colon B\,\mathrm{Aut}_{\mathbf{EqBr}}(\mathcal{A}) \to B^4\mathbf{k}^\times$.

For any monoidal functor $\rho\colon G \to \mathrm{Aut}_{\mathbf{EqBr}}(\mathcal{A})$, $\mathbf{WRT}_{\mathcal{A}}$ obtains a $(\sigma^4_{BG,\rho^*c}, \rho)$-module structure on $\mathbf{WRT}_{\mathcal{A}}$. The question of trivializing the anomaly is the question of σ^4_{BG,ρ^*c} having a Neumann boundary theory, which is equivalent to whether or not c pulls back to define the trivial cohomology class on BG.

So there are many different four-dimensional theories in the story: the untwisted pure topological G-gauge theory (which we *want* to be defined relative to, in order to gauge), and then the Crane-Yetter theory and the twisted topological G-gauge theory (which we are *always* on the boundary of).

One way to think of the universal such theory $\sigma_{B\,\mathrm{Aut}_{\mathbf{EqBr}},c}$ is as the result of gauging the canonical anomaly-free G-action on $\mathbf{CY}_{\mathcal{A}}$ defined by ρ.

1.6. **Gauging and anomalies.** Given a quantum field theory F and a (compact Lie) group G, under certain conditions, a new theory, the *G-gauged theory*, can be produced via a procedure known as *gauging*. This has received various mathematical formulations, e.g. for topological theories see [Tel14, §2.3]. Also see [FMT22, §3.4]. For non-extended versions of some of the 3-dimensional theories considered in this paper, see [Mül20, MS20]. This procedure cannot always be carried out, and the obstructions are known as *'t Hooft anomalies* [tH80].

Gauging is often phrased as a two step process:

(1) couple F to a background G-field (principal G-bundle),
(2) integrate over all G-bundles to obtain a new ordinary theory: the gauged theory.

In the context of topological field theories, the first step is to extend[6] F from $\mathbf{Bord}^{\mathrm{fr}}_d$ (or e.g. $\mathbf{Bord}^{\mathrm{or}}_d$) to $\mathbf{Bord}^{BG}_d$.

Remark 1.10. Even before proceeding to the second step, one can see how anomalies of the sort defined in Section 3.2 can act as obstructions to gauging. One should imagine that we cannot quite compatibly extend F to $\mathbf{Bord}^{BG}_d$: We can only extend it as a relative theory. I.e. we have a (once-categorified) theory α (which *is* defined on $\mathbf{Bord}^{BG}_d$) and a relative theory $F\colon \alpha \to 1$. This becomes a TQFT defined on $\mathbf{Bord}^{BG}_d$ only once we provide a trivialization $1 \xrightarrow{\sim} \alpha$.

As in Section 1.5, a theory defined on $\mathbf{Bord}^{BG}_d$ defines a boundary theory $\widetilde{F}\colon 1 \to \sigma^{d+1}_{BG}$. (see Section B.1 for notation, and Section 3.2 for a more detailed discussion).

The second step is easier to formulate in this language. Namely, as in [FMT22, Definition 3.22, Example 3.24], the process of integrating over all G-bundles is pairing with the *Neumann boundary*

[6]There is always a trivial extension of F to $\mathbf{Bord}^{BG}_d$. The pullback of any extension along $\mathbf{Bord}_d \to \mathbf{Bord}^{BG}_d$ (sending every bordism to the same bordism equipped with the product principal G-bundle) is an ordinary theory with internal G-symmetry. Usually one has a given (internal) action of G on F, and one insists that the extension agrees with this action after pulling back.

theory:

$$F/_\epsilon \sigma \coloneqq \epsilon \otimes_{\sigma^d_{BG}} \widetilde{F} \ , \tag{1.11}$$

where the Neumann boundary theory ϵ is the morphism induced (as in Proposition B.12) by the augmentation:

$$\begin{array}{ccc} & BG & \\ \swarrow & & \searrow \\ BG & & * \end{array}$$

Note that if the theory σ^{d+1}_{BG} is twisted by a cocycle τ, then we will not necessarily have an augmentation map: We need to trivialize τ. I.e. if we have a boundary theory

$$\widetilde{F} \colon 1 \to \sigma^{d+1}_{BG,\tau} \ ,$$

in order to gauge the G-action, we need to pair with an augmentation for BG. One obtains an augmentation from any correspondence of the form

$$\begin{array}{ccc} & (BH,\mu) & \\ \swarrow & & \searrow \\ (BG,\tau) & & * \end{array}$$

where $H \subset G$ is a subgroup, and μ is a trivialization of $\tau|_H$. This is, for example, the form of the classification of simple, fully extended 2-dimensional topological theories relative to gauge theory in [FT22]: subgroups equipped with central extensions.

As is explained in Section 1.5, the cocycle τ classifies an anomaly theory α_τ. So a trivialization of α_τ (i.e. of τ) determines an augmentation, which induces the Neumann boundary condition, which allows us to gauge as in (1.11). In other words: α_τ obstructs gauging.

2. Projective TQFTs

2.1. **Preliminaries: arrow categories.** Recall the arrow category $\mathcal{T}^{\downarrow}$ defined in [JFS17]. An anomalous theory $\alpha \to 1$ is, by definition (see Remark A.11), a functor

$$\overline{Z} \colon \mathbf{Bord}_d \to \mathcal{T}^{\downarrow} \tag{2.1}$$

such that

$$s \circ \overline{Z} = \alpha \qquad\qquad t \circ \overline{Z} = 1 \ .$$

Remark 2.2. As explained in Remark A.11, the relative theories in this paper are lax natural transformations. Everything can be repeated to produce an oplax version by replacing $\mathcal{T}^{\downarrow}$ with $\mathcal{T}^{\rightarrow}$ everywhere.

Recall that [JFS17] define the arrow category 'terminating at the unit' to be:

$$\mathcal{T}^{\downarrow 1} \coloneqq \mathcal{T}^{\downarrow} \underset{\mathcal{T}}{\overset{h}{\times}} \mathcal{T}[0] \qquad\qquad \begin{array}{ccc} \mathcal{T}^{\downarrow 1} & \longrightarrow & \mathcal{T}[0] \\ \downarrow & & \downarrow \\ \mathcal{T}^{\downarrow} & \xrightarrow{s} & \mathcal{T} \end{array} \ .$$

Similarly, we will define the subcategory 'originating at an invertible object' to be:

$$\mathcal{T}^{\times\downarrow} \coloneqq \mathcal{T}^{\times} \underset{\mathcal{T}}{\overset{h}{\times}} \mathcal{T}^{\downarrow} \qquad\qquad \begin{array}{ccc} \mathcal{T}^{\times\downarrow} & \longrightarrow & \mathcal{T}^{\downarrow} \\ \downarrow & & \downarrow t \\ \mathcal{T}^{\times} & \longrightarrow & \mathcal{T} \end{array} \ .$$

2.2. **The projectivization of a target category.** The projectivization of the category $\Omega\mathcal{T}$, written $\mathbb{P}\Omega\mathcal{T}$, consists of arrows in $\mathcal{T}$ between an invertible object of $\mathcal{T}$ and the unit in $\mathcal{T}$. For instance, for $\mathcal{T} = \mathbf{Alg}$ an object is a module over an Azumaya algebra, which up to Morita equivalence is a vector space defined only up to scalar; see Section 2.3.2.

Definition 2.3. Define the *projectivization* of a symmetric monoidal (∞, n)-category (thought of as an $\mathbb{E}_\infty$-algebra among complete n-fold Segal spaces) $\mathcal{T}$ to be:

$$\mathbb{P}(\Omega\mathcal{T}) \coloneqq \mathcal{T}[0] \overset{h}{\underset{\mathcal{T}}{\times}} \mathcal{T}^{\downarrow} \overset{h}{\underset{\mathcal{T}}{\times}} \mathcal{T}^{\times}$$

Recall from [JFS17, Def. 6.10] that $\mathcal{T}[0] \simeq *$.

Warning 1. This construction does *not* only depend on $\Omega\mathcal{T}$, as the notation might suggest. It depends on the group of isomorphism classes of invertible objects of $\mathcal{T}$. For example, many categories are of the form $\Omega\mathcal{T}_1 \simeq \Omega\mathcal{T}_2$ where $\pi_0\mathcal{T}_1^{\times}$ is trivial and $\pi_0\mathcal{T}_2^{\times}$ is not. We nonetheless keep the notation $\mathbb{P}\Omega\mathcal{T}$, which should always be read as a function of $\mathcal{T}$.

This becomes especially important when we study 3-dimensional projective targets in Section 2.3.3. See Remark 2.16.

Theorem 2.4. *There is a well-defined functor from the ∞-groupoid of symmetric monoidal (∞, n)-categories (and equivalences) to itself,*

$$\mathbb{P}\colon \mathbb{E}_\infty(n\text{-}\mathbf{cat}) \to \mathbb{E}_\infty(n\text{-}\mathbf{cat})$$

which sends a symmetric monoidal (∞, n)-category $\mathcal{T}$ to $\mathbb{P}\Omega\mathcal{T}$, thought of as $\mathbb{E}_\infty$-algebras among complete n-fold Segal spaces.

Furthermore, projectivization commutes with iterated looping, i.e. the following diagram commutes for all integers $j \in \mathbb{Z}$ satisfying $1 \le j \le n$:

$$\begin{array}{ccc} \mathbb{E}_\infty(n\text{-}\mathbf{cat}) & \xrightarrow{\mathbb{P}} & \mathbb{E}_\infty(n\text{-}\mathbf{cat}) \\ \downarrow{\scriptstyle \Omega^j} & & \downarrow{\scriptstyle \Omega^j} \\ \mathbb{E}_\infty((n-j)\text{-}\mathbf{cat}) & \xrightarrow{\mathbb{P}} & \mathbb{E}_\infty((n-j)\text{-}\mathbf{cat}) \end{array} \tag{2.5}$$

Proof. The functor $(-)^{\downarrow}$ is defined in [JFS17, Definition 5.14], and shown to depend naturally on both arguments in [JFS17, Corollary 5.19], which is extended to the symmetric monoidal case in [JFS17, Corollary 6.9]. Rephrasing this slightly, this says that $(-)^{\downarrow}$ defines a functor from the ∞-groupoid of symmetric monoidal (∞, n)-categories and equivalences to itself.

There is a commuting diagram of three homotopy fiber squares:

$$\begin{array}{ccccc} \mathbb{P}\Omega\mathcal{T} & \longrightarrow & \mathcal{T}^{\downarrow 1} & \xrightarrow{s} & \mathcal{T}[0] \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{T}^{\times\downarrow} & \longrightarrow & \mathcal{T}^{\downarrow} & \xrightarrow{s} & \mathcal{T} \\ \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle t} & & \\ \mathcal{T}^{\times} & \longrightarrow & \mathcal{T} & & \end{array} \tag{2.6}$$

because

$$\left(\mathcal{T}^{\downarrow 1}\right) \overset{h}{\underset{\mathcal{T}^{\downarrow}}{\times}} \left(\mathcal{T}^{\times\downarrow}\right) \simeq \mathcal{T}[0] \overset{h}{\underset{\mathcal{T}}{\times}} \mathcal{T}^{\downarrow} \overset{h}{\underset{\mathcal{T}}{\times}} \mathcal{T}^{\times} \ .$$

Therefore the functoriality of $\mathbb{P}$ follows from the functoriality of $(-)^{\downarrow}$ [JFS17] and the functoriality of $(-)^{\times}$.

For a fixed non-negative integer n, we will first show the statement for $k = 1$:

$$\Omega\mathbb{P}\Omega(-) \simeq \mathbb{P}\Omega^2(-) \ . \tag{2.7}$$

Recall that, by definition (here $\mathbb{P}\Omega$ is applied to $\mathcal{T}$, and to $\Omega\mathcal{T}$ respectively):

$$\Omega\mathbb{P}\Omega\mathcal{T} = \Omega\left(\mathcal{T}\left[0\right] \underset{\mathcal{T}}{\overset{h}{\times}} \mathcal{T}^{\downarrow} \underset{\mathcal{T}}{\overset{h}{\times}} \mathcal{T}^{\times}\right)$$

and

$$\mathbb{P}\Omega^2\mathcal{T} = \Omega\mathcal{T}\left[0\right] \underset{\Omega\mathcal{T}}{\overset{h}{\times}} (\Omega\mathcal{T})^{\downarrow} \underset{\Omega\mathcal{T}}{\overset{h}{\times}} (\Omega\mathcal{T})^{\times} \ .$$

There is a canonical equivalence of symmetric monoidal complete n-fold Segal spaces ([JFS17, Proposition 6.12]):

$$\Omega\mathcal{T} \simeq \mathcal{T}\left[0\right] \underset{\mathcal{T}}{\overset{h}{\times}} \mathcal{T}^{\downarrow} \underset{\mathcal{T}}{\overset{h}{\times}} \mathcal{T}\left[0\right] \ ,$$

which means that (2.7) follows from the fact that homotopy limits commute with each other, together with the facts that Ω commutes with $(-)^{\downarrow}$ and with $(-)^{\times}$.

To see the statement for general $j \in \{1, \ldots, n\}$, apply (2.7) to $\Omega^{j-1}\mathcal{T}$. □

Remark 2.8. The $\mathbb{P}$ in Definition 2.3 can be thought of as standing for "projective", as $\mathbb{P}\mathbf{Vect}$ should be thought of as some version of the category of projective spaces. Therefore it can also be thought of as standing for "pure states", as these form the projectivization of the Hilbert space of mixed states.

As is conjectured by Freed in [Fre23], the projectivization should have a description as a directed pullback of symmetric monoidal (∞, n)-categories

$$\begin{array}{ccc} \mathbb{P}\Omega\mathcal{T} & \overset{F}{\dashrightarrow} & \mathcal{T}^{\times} \\ \downarrow & & \downarrow \\ * & \longrightarrow & \mathcal{T} \end{array} \tag{2.9}$$

One way to interpret (2.9), is as expressing "exactness" at a term of an "exact sequence" of symmetric monoidal ∞-categories. We do not attempt to make this notion precise here, however consider the following diagram of functors:

$$\begin{array}{ccccccc} \cdots & \Omega\mathcal{T}^{\times} & \longrightarrow & \Omega\mathcal{T} & \longrightarrow & \mathbb{P}\Omega\mathcal{T} & \\ & \mathcal{T}^{\times} & \longrightarrow & \mathcal{T} & \longrightarrow & \mathbb{P}\mathcal{T} & \cdots \end{array} \tag{2.10}$$

where the connecting homomorphism is F from (2.9). The functors $\Omega\mathcal{T} \to \mathbb{P}\Omega\mathcal{T}$ and $\mathcal{T} \to \mathbb{P}\mathcal{T}$ in (2.9) are the canonically defined functors which send an object to the same object regarded as an endomorphism of the unit. We hope to return to this in future work.

Remark 2.11. Note that, if every invertible object of $\mathcal{T}$ is trivializable, then the functor $\Omega\mathcal{T} \to \mathbb{P}\Omega\mathcal{T}$ is essentially surjective. Otherwise, there are objects in $\mathbb{P}\Omega\mathcal{T}$ outside of the image of $\Omega\mathcal{T}$: for example, if $\mathcal{T}$ consists of algebra objects in $\Omega\mathcal{T}$, consider any nontrivial invertible object of $\mathcal{T}$. Then the regular module defines an object of $\mathbb{P}\Omega\mathcal{T}$ which is not in the image of $\Omega\mathcal{T}$.

2.3. **Fully-extended projective target categories.**

2.3.1. *Projective Morita categories.* As discussed in Warning 1, the construction of $\mathbb{P}\left(\Omega\mathcal{T}\right)$ actually depends on $\mathcal{T}$, rather than merely $\Omega\mathcal{T}$. However, if we are given a symmetric monoidal ($\otimes$-sifted-cocomplete) (∞, n)-category $\mathcal{S}$, we can form a projectivization of $\mathcal{S}$ as follows (see Remark 2.12 for a discussion of different models of the higher Morita category). We we can form the symmetric monoidal $(\infty, n+1)$-category $\mathcal{T} = \mathbf{Alg}\left(\mathcal{S}\right)$, the "even higher" Morita category of $\mathbb{E}_1$-algebras [JFS17].

Recall that $\Omega\mathcal{T} \simeq \mathcal{S}$, so in this case we obtain a motto about projective objects: *An object of $\mathbb{P}\left(\mathcal{S}\right)$ can be thought of as an invertible (e.g. Azumaya) algebra in $\mathcal{S}$ along with a module over this algebra (also internal to $\mathcal{S}$).*

More generally, we obtain a projective target category by taking the projectivization of the Morita $(\infty, n+k)$-category $\mathbf{Alg}_n(\mathcal{S})$ for a (sufficiently nice) symmetric monoidal (∞, k)-category $\mathcal{S}$.

Remark 2.12 (Models for Morita categories). The higher Morita (∞, n)-category of $\mathbb{E}_n$-algebras in a $\otimes$-sifted-cocomplete (resp. $\otimes$-GR-cocomplete) symmetric monoidal (∞, k)-category $\mathcal{S}$ was constructed in [Sch14] (resp. [Hau17]). This was extended to the "even higher" Morita $(\infty, n+k)$-category, written $\mathbf{Alg}_n\mathcal{S}$, in [JFS17].

There are expected relationships between the models in [Sch14] and [Hau17], but the author is unaware of a theorem relating them. One benefit of the former is that n-dualizability is known [GS18, Theorem 5.1][7], but the arguments do not translate to the model in [Hau17].

On the other hand, the results used in Section 5.1 concerning the dualizability and invertibility of (braided) fusion categories are from [BJS21, BJSS21], which use the model from [Hau17].

2.3.2. *The projectivization of the category of vector spaces.* Let $\mathcal{S} = \mathbf{Vect}$ be the symmetric monoidal category of finite-dimensional complex vector spaces. Setting $\mathcal{T} = \mathbf{Alg}$ to be the symmetric monoidal Morita 2-category of associative algebras, we can calculate $\mathbb{P}\mathbf{Vect}$ to be the 2-category given by "all modules over all Azumaya algebras". The 1-morphisms are (op)lax squares

$$\begin{array}{ccc} 1 & = & 1 \\ {\scriptstyle {}_A M}\downarrow & \overset{\varphi}{\nearrow} & \downarrow{\scriptstyle {}_B N} \\ A & \xrightarrow{\sim} & B \end{array}$$

where A and B are Morita-invertible algebras with modules M and N. The data of the filling φ is a linear map between M and N which intertwines the module structures according to an invertible Morita morphism $A \xrightarrow{\sim} B$. The following are examples of objects of $\mathbb{P}\mathbf{Vect}$.

- Any fully-dualizable (i.e. separable) algebra as a module over itself.
- If there is an algebra map to the trivial algebra (an augmentation map) then this defines a module structure on the one-dimensional vector space.
- The rank n vector space as a module over the matrix algebra M_n.

Remark 2.13. Note that $\mathbb{P}\mathbf{Vect}$ serves as a target for non-extended projective TQFTs in any dimension. The category $\mathbb{P}\mathbf{Vect}$ is used in [Haï25] to formalize the notion of anomalous non-extended non-semisimple Witten-Reshetikhin-Turaev theory.

Remark 2.14. There is a related 2-category, written **Proj**, which is defined in [Fre23, Appendix A]. The objects are finite-dimensional vector spaces, the 1-morphisms are linear maps, and there is a 2-morphism between any two linear maps $\varphi, \psi \colon V \to W$ for every $\lambda \in \mathbb{C}^\times$ such that $\varphi(v) = \lambda\psi(v)$ for all $v \in V$.

There is an equivalence $\mathbf{Proj} \to \mathbb{P}\mathbf{Vect}$, e.g. as symmetric monoidal bicategories ([Haï25, Remark 7.2]), which sends a vector space V to itself regarded as an endomorphism of the unit in **Alg**, i.e. as a $\mathbb{C}$-bimodule.

2.3.3. *The projectivization of the Morita category of monoidal categories.* Let $\mathcal{T} = \mathbf{BrFus}$ be the Morita 4-category of braided fusion categories (see Section 4.1) and recall:

$$\mathbf{Fus} \simeq \Omega\mathcal{T} = \mathrm{End}_{\mathcal{T}}(1) \ .$$

Applying $\mathbb{P}$ from Definition 2.3 to **BrFus**, we obtain the following.

Corollary 2.15. $\mathbb{P}\mathbf{Fus}$ *is a symmetric monoidal* 4*-category. The objects of* $\mathbb{P}\mathbf{Fus}$ *are arrows in* **BrFus** *which are between an invertible object and the unit (i.e. central modules over invertible*

[7] Note however that the only $(n+1)$-dualizable object is the unit [GS18, Theorem 6.1].

braided fusion categories). The 1*-morphisms in* $\mathbb{P}$**Fus** *are (op)lax squares:*

$$\begin{array}{ccc} 1 & = & 1 \\ {\scriptstyle {}_{\mathcal{A}}\mathcal{M}}\downarrow & \overset{\Phi}{\nearrow} & \downarrow{\scriptstyle {}_{\mathcal{B}}\mathcal{N}} \\ \mathcal{A} & \xrightarrow{\sim} & \mathcal{B} \end{array}$$

where $\mathcal{A}$ *and* $\mathcal{B}$ *are invertible objects of* **BrFus***,* $\mathcal{M}$ *and* $\mathcal{N}$ *are (possibly non-invertible)* 1*-morphisms in* **BrFus***, and the filling* Φ *is a* 2*-morphism in* **BrFus***, i.e. a 'twisted' intertwining bimodule between* $\mathcal{M}$ *and* $\mathcal{N}$*. The higher morphisms are defined by general principles in* [JFS17].

Remark 2.16. Recall from Warning 1 that the construction of $\mathbb{P}\Omega\mathcal{T}$ actually depends on the group of isomorphism classes of invertible objects of $\mathcal{T}$, rather than just $\Omega\mathcal{T}$. In low dimensions this didn't appear, unless we were considering super-vector spaces, because the category of finite-dimensional vector spaces, the Morita category of algebras, and the Morita category of fusion categories all have trivial group of isomorphism classes of invertible objects, i.e. all invertible objects are trivializable.

This is not true in this dimension: The Morita 4-category **BrFus** has a highly nontrivial group $\pi_0\mathbf{BrFus}^\times$, which is known to be equivalent to the so-called Witt group of braided fusion categories [BJSS21, Theorem 4.2]. Note that this contains the Witt group of $\mathbb{Q}$, and is therefore certainly nontrivial. This clearly produces a different projectivization in comparison with performing the same construction, say, in the arrow category of the 4-category B**Fus**, which would clearly not detect any of the non-trivializable invertible objects of **BrFus**.

2.4. **Projective theories.**

Definition 2.17. A d-dimensional *projective* (X,ζ)*-TQFT* is a symmetric monoidal functor:

$$\overline{F}\colon \mathbf{Bord}_d^{(X,\zeta)} \to \mathbb{P}\Omega\mathcal{T}\ .$$

Write $s\colon \mathbb{P}\Omega\mathcal{T} \to \mathcal{T}^\times$ for the composition of the vertical functors on the left in (2.6). Given such a theory $\overline{F}$, composing with the functor s results in an invertible theory:

$$\alpha\colon \mathbf{Bord}_d^{(X,\zeta)} \xrightarrow{\overline{F}} \mathbb{P}\Omega\mathcal{T} \xrightarrow{s} \mathcal{T}^\times\ .$$

This theory is the *projectivity* of the projective theory $\overline{F}$. Sometimes to emphasize that a theory is *not* projective, we will call it *linear*.

Proposition 2.18. *A trivialization of the projectivity* $\alpha = s \circ \overline{F}$ *determines a linear theory* $F_X\colon \mathbf{Bord}_d^X \to \Omega\mathcal{T}$*. We will call* F_X *a* linearization *of* $\overline{F}$*.*

Proof. A projective TQFT is equivalently a natural transformation from an invertible TQFT to the trivial TQFT. Composing this with the trivialization results in an endomorphism of the trivial theory valued in $\mathcal{T}$ which, by Proposition A.9, is equivalent to a TQFT valued in $\Omega\mathcal{T}$. □

Definition 2.19. Assume a projective TQFT $\overline{F}$ has a linearization upon restriction to framed bordisms, i.e. there is a theory F^{fr} such that the following diagram commutes:

$$\begin{array}{ccccc} \mathbf{Bord}_d^{\mathrm{fr}} & \hookrightarrow & \mathbf{Bord}_d^X & & \\ \downarrow{\scriptstyle F^{\mathrm{fr}}} & \searrow^{\overline{F}^{\mathrm{fr}}} & \downarrow{\scriptstyle \overline{F}} & \searrow^{\alpha} & \\ \Omega\mathcal{T} & \longrightarrow & \mathbb{P}\Omega\mathcal{T} & \longrightarrow & \mathcal{T}^\times \end{array} \tag{2.20}$$

Then we call F^{fr} the *underlying framed theory of* $\overline{F}$.

2.5. **Extensions of the bordism category.** Just as in classical representation theory, a projective representation is equivalent to a linear representation of an extension. The notion of an extension of a bordism category by a modular functor already appeared in Segal's original paper [Seg04, Definition 5.2].

The projectivity of a projective TQFT in the sense of Definition 2.17 is also captured by an extension of a bordism category as follows. Namely, the extension is the universal object over which the projectivity can be trivialized. In slightly more detail, it is the pullback:

$$\begin{array}{ccccc} \widetilde{\mathbf{Bord}}_d^X & \longrightarrow & \mathbf{Bord}_d^X & & \\ \downarrow{\scriptstyle \widetilde{F}} & & \downarrow{\scriptstyle \overline{F}} & \searrow^{\alpha} & \\ \Omega\mathcal{T} & \longrightarrow & \mathbb{P}\Omega\mathcal{T} & \longrightarrow & \mathcal{T}^\times \end{array} \tag{2.21}$$

and since the composition $\Omega\mathcal{T} \to \mathbb{P}\Omega\mathcal{T} \to \mathcal{T}^\times$ is canonically trivialized, so is the projectivity of $\widetilde{F}$, i.e. the extension is the universal object over which the projectivity is trivializable.

This description as a pullback of symmetric monoidal ∞-categories is not completely precise. A more formal description is as follows: Given a projective TQFT $\overline{F}\colon \mathbf{Bord}_d^X \to \mathbb{P}\Omega\mathcal{T}$, the projectivity $\alpha = s \circ \overline{F}$ is an invertible TQFT, and therefore classifies a map of groupoids $\widetilde{\alpha}$ (A.14) as explained in Section A.3. The homotopy fiber $\mathbf{E}$ of $\widetilde{\alpha}$ is some extension of the (groupoid given by invertible all morphisms in the) original bordism category:

$$\begin{array}{ccccc} \Omega\mathcal{T}^\times & \longrightarrow & \mathbf{E} & \longrightarrow & * \\ \downarrow & & \downarrow & & \downarrow \\ * & \longrightarrow & \left|\mathbf{Bord}_d^X\right| & \xrightarrow{\alpha} & \mathcal{T}^\times \end{array}$$

and the claim is that $\mathbf{E}$ is the result of inverting all morphisms in some extension of the original bordism category.

Remark 2.22. Recall from Remark 2.11 that the map $\Omega\mathcal{T} \to \mathbb{P}\Omega\mathcal{T}$ might not be essentially surjective, and therefore the map from $\widetilde{\mathbf{Bord}}_d^X$ to $\mathbf{Bord}_d^X$ might not be essentially surjective either.

One instance of this appearing in examples is the filled bordism category, where bordisms (of any codimension) are equipped with a filling, i.e. a bordism of one higher dimension of which they are the boundary (so in particular these fillings are manifolds with corners). For example, the point is not an object of the fully-extended filled bordism category, but S^0 is.

Add a reference for the filled bordism category if one exists (Walker? Monnier?).

The bordism category $\mathbf{Bord}_{d+1}^{\widetilde{X}}$ provides a description for (a fully-extended version of) the extension of the source discussed in [Fre23, §3]. See also Remark 3.10.

3. Anomalies

See [Fre23] and the references therein for more details on anomalies.

3.1. **Definitions.** Fix the target $\mathcal{T}$ from Section A.1. Let (X, ζ) be as in Section A.1.1, and recall we have defined (X, ζ)-theories as functors out of $\mathbf{Bord}^{(X,\zeta)}$.

Definition 3.1. An *anomaly* α is an invertible, once-categorified TQFT:

$$\alpha\colon \mathbf{Bord}_d^{(X,\zeta)} \to \mathcal{T}\ .$$

A *d-dimensional anomalous TQFT F with anomaly α* is a TQFT defined relative to α:

$$F_\alpha\colon \alpha \to 1\ ,$$

where 1 denotes the trivial theory on $\mathbf{Bord}_d^{X,\zeta}$.

A *trivialization of the anomaly* α is an equivalence $1 \xrightarrow{\sim} \alpha$. The *trivialized theory* is the composition:

$$1 \xrightarrow{\sim} \alpha \xrightarrow{F} 1$$

which defines a (X,ζ)-TQFT of dimension d by Proposition A.9.

Remark 3.2. Sometimes anomalies are defined to be boundary theories rather than relative theories (see Remark A.13).

Sometimes anomalies are defined as left boundary/relative theories $1 \to \alpha$, rather than right boundary/relative theories: $\alpha \to 1$. In this case trivializations would be $\alpha \xrightarrow{\sim} 1$.

Given an (X,ζ)-theory Z, write Z_{fr} for the underlying framed theory given by restricting Z to bordisms with trivial (X,ζ)-structure.

Definition 3.3. Consider two tangential structures (X_0,ζ_0) and (X,ζ) such that there is a map $X \to X_0$ compatible with ζ_0 and ζ. We say a TQFT:

$$F\colon \mathbf{Bord}_d^{X_0} \to \Omega\mathcal{T}$$

has an anomaly as an (X,ζ)*-theory* if there is an anomaly theory

$$\alpha\colon \mathbf{Bord}_d^{(X,\zeta)} \to \mathcal{T}$$

and a trivialization $t\colon 1_{\mathrm{fr}} \xrightarrow{\sim} \alpha_{\mathrm{fr}}$, along with an anomalous theory

$$F_\alpha\colon \alpha \to 1_{(X,\zeta)}$$

(where $1_{(X,\zeta)}$ denotes the trivial (X,ζ)-theory) and an equivalence:

$$\theta\colon F \simeq F_\alpha \circ t\ .$$

Definition 3.4. An (X,ζ)*-anomaly of the theory* F is such a quadruple $(\alpha, F_\alpha, t, \theta)$. The trivialization t and equivalence θ are often canonically defined, in which case we will write that $\alpha \xrightarrow{F_\alpha} 1_X$ is the (X,ζ)-anomaly of F.

Proposition 3.5. *A projective theory with projectivity* α *(in the sense of Definition 2.17) is naturally equivalent to an anomalous TQFT with anomaly* α *(in the sense of Definition 3.3).*

Proof. An anomalous TQFT is a natural transformation from an invertible TQFT to the unit. This is, by definition, a functor from the bordism category to the arrow category such that composition with the source and target functors give you an invertible object and the unit respectively. This is precisely the same condition as factoring through $\mathbb{P}\Omega\mathcal{T} \to \mathcal{T}^{\downarrow}$, i.e. defining a projective TQFT. □

Example 3.6 (Framing anomaly). Let $(X,\zeta) = (B\,\mathrm{SO}\,(d)\,,\zeta_\iota)$ be as in Example A.5, i.e. $\mathbf{Bord}_d^{(X,\zeta)} \simeq \mathbf{Bord}_d^{\mathrm{or}}$. A framed theory F is often said to have a nontrivial *framing anomaly*, if it cannot be upgraded to an oriented theory, i.e. a $(B\,\mathrm{SO}\,(d)\,,\zeta_\iota)$-theory. This is equivalent to having a $(B\,\mathrm{SO}\,(d)\,,\zeta_\iota)$-anomaly as in Definition 3.4 such that the associated anomaly theory α is a non-trivial theory.

Example 3.7. Let $(X,\zeta) = (BG,\zeta_{\mathrm{triv}})$ be as in Example A.7. A BG-anomaly as in Definition 3.4 (or G-anomaly) for the theory F is an anomalous action of G on F. We will see in Theorem 3.12 that this is equivalent to F having a $\left(\sigma_{BG,\tau}^{d+1},\rho\right)$-module structure (Section B.2) where the twist τ is closely related to the anomaly theory.

3.2. **The anomaly associated to a projectivity class.** Let X be a space (higher groupoid) equipped with a cocycle c classifying $\widetilde{X}$:

$$\begin{array}{c} \widetilde{X} \\ \downarrow{\scriptstyle \pi} \\ X \xrightarrow{c} B^{d+1}\mathbf{k}^\times \end{array}$$

Let $\Omega\mathcal{T} = \mathrm{End}_{\mathcal{T}}(1)$ denote the looping of the fixed target $\mathcal{T}$ from the beginning of Section A.1. The anomaly theory α_c is defined as follows. By the Cobordism Hypothesis (A.6), an X-theory $\mathbf{Bord}^X_{d+1} \to \mathcal{T}$ is determined by a functor from X to $\mathcal{T}^\sim$, so we can define α_c to be the restriction to $\mathbf{Bord}^X_d$ of the theory classified by the following composition of functors:

$$X \xrightarrow{c} B^{d+1}\mathbf{k}^\times \hookrightarrow \mathcal{T} \ .$$

By definition, α_c factors through "scalars" in the target, which is sufficient to be an invertible TQFT. I.e. a relative theory $\alpha_c \to 1_X$ is an anomalous X-theory with anomaly α_c.

Proposition 3.8. *An anomalous X-theory $F_\alpha \colon \alpha_c \to 1_X$ determines an $\widetilde{X}$-theory $\widetilde{F} \colon \mathbf{Bord}^{\widetilde{X}}_d \to \Omega\mathcal{T}$.*

Proof. By the Cobordism Hypothesis (A.6) applied to the arrow category $\mathcal{T}^{\downarrow}$ (see [JFS17, Theorem 7.15]), the relative theory F_α is classified by a functor from X to $\left(\mathcal{T}^{\downarrow}\right)^\sim$ with source α_c and target 1, i.e. by a map of bundles over X:

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{|F_\alpha|} & X \times \Omega\mathcal{T}^\sim \\ & {\scriptstyle \pi}\searrow \quad \swarrow{\scriptstyle p_1} & \\ & X & \end{array}$$

Composing with the projection to the second factor we obtain a map

$$\widetilde{X} \xrightarrow{|F_\alpha|} X \times \Omega\mathcal{T}^\sim \xrightarrow{p_2} \Omega\mathcal{T}^\sim \ .$$

The result follows from the Cobordism Hypothesis (A.6): Every TQFT $\widetilde{F} \colon \mathbf{Bord}^{\widetilde{X}}_d \to \Omega\mathcal{T}$ is classified by such a map $\widetilde{X} \to \Omega\mathcal{T}^\sim$. □

Example 3.9. Fix $d = 1$ and let c be a cocycle classifying a central extension $\widetilde{G}$ of a group G by $\mathbf{k}^\times$. If $X = BG$ and $\widetilde{X} = B\widetilde{G}$, then a projective representation of G with projectivity cocycle c (i.e. a linear representation of $\widetilde{G}$) defines a such a theory $\widetilde{F}$.

Remark 3.10. Recall that the upshot of the discussion in Section 2.5 is that every projective TQFT is an honest (i.e. non-anomalous) functor out of an extension of the original bordism category. According to Proposition 3.5, the same is true for anomalous theories. This is the familiar statement that equipping your bordisms with extra structure allows one to trivialize the anomaly. See Remark 5.6 for more on this extended category in the context of Reshetikhin-Turaev and Crane-Yetter TQFTs.

The bordism category $\mathbf{Bord}^{\widetilde{X}}_{d+1}$ provides a description for (a fully-extended version of) the extended bordism category for 't Hooft anomalies described by such a cocycle c.

Note that, as discussed in Remark 2.22, the functor from the extended bordism category to the original one is not necessarily essentially surjective, e.g. it might consist e.g. of "filled" bordisms if the anomalous theory in question does not extend to the point. For instance, in this case the point is not an object of the extended category, but S^0 is.

3.2.1. *Trivializing the anomaly.* We have a symmetric monoidal functor $\pi_*\colon \mathbf{Bord}_d^{\widetilde{X}} \to \mathbf{Bord}_d^X$ given by composing the map defining the $\widetilde{X}$-structure on a bordism with the map π.

Question 1. Does $\widetilde{F}$ factor as follows?

$$\begin{array}{ccc} \mathbf{Bord}_d^{\widetilde{X}} & \xrightarrow{\widetilde{F}} & \Omega\mathcal{T} \\ \downarrow{\scriptstyle \pi_*} & \nearrow_{F_X} & \\ \mathbf{Bord}_d^X & & \end{array}$$

Question 1 is equivalent to the question of splitting $\widetilde{X} \to X$ (i.e. trivializing c). We can ask a weaker question, by pulling back along a map $f\colon Y \to X$. Write $f^*\widetilde{X}$ for the pullback:

$$\begin{array}{ccc} f^*\widetilde{X} & \longrightarrow & \widetilde{X} \\ \downarrow{\scriptstyle f^*\pi} & & \downarrow{\scriptstyle \pi} \\ Y & \xrightarrow{f} & X \end{array}$$

Write $f^*\widetilde{F}$ for the corresponding $f^*\widetilde{X}$-theory:

$$f^*\widetilde{F}\colon \mathbf{Bord}_d^{f^*\widetilde{X}} \to \mathbf{Bord}_d^{\widetilde{X}} \xrightarrow{\widetilde{F}} \Omega\mathcal{T}\ .$$

Now we can state a refinement of Question 1.

Question 2. Does $f^*\widetilde{F}$ factor as follows?

$$\begin{array}{ccc} \mathbf{Bord}_d^{f^*\widetilde{X}} & \xrightarrow{f^*\widetilde{F}} & \Omega\mathcal{T} \\ \downarrow{\scriptstyle (f^*\pi)_*} & \nearrow_{F_Y} & \\ \mathbf{Bord}_d^Y & & \end{array}$$

Question 2 is similarly equivalent to splitting $f^*\widetilde{X} \to Y$, or equivalently trivializing f^*c.

A trivialization $1 \xrightarrow{\sim} \alpha_c$ would provide us with an X-theory by Proposition A.9. However if c is not trivializable, the best we will be able to do is to trivialize the pullback along some map $f\colon Y \to X$. This allows us to regard the anomaly theory α_c as a Y-theory:

$$\begin{array}{ccc} \mathbf{Bord}_d^Y & & \\ \downarrow{\scriptstyle f} & \searrow^{\alpha_{f^*c}} & \\ \mathbf{Bord}_d^X & \xrightarrow{\alpha_c} & \mathcal{T} \end{array} \tag{3.11}$$

A trivialization of $\widetilde{X}$ over Y (i.e. of f^*c) uniquely determines a trivialization $1_Y \xrightarrow{\sim} \alpha_c^Y$. Composing with the anomalous theory gives $1_Y \xrightarrow{\sim} \alpha_c^Y \to 1_Y$, which is a Y-theory by Proposition A.9.

3.3. **Anomalies and sandwiches.** Let $\mathcal{T}$ be the $(d+1)$-dimensional target fixed in Section A.1. The d-dimensional TQFTs we will consider will be valued in the looping $\Omega\mathcal{T}$. Let X be a (pointed, connected) π-finite space. Consider a cocycle $c\colon X \to B^{d+1}\mathbf{k}^\times$ classifying $\widetilde{X}$:

$$\begin{array}{ccc} \widetilde{X} & & \\ \downarrow{\scriptstyle \pi} & & \\ X & \xrightarrow{c} & B^{d+1}\mathbf{k}^\times\ . \end{array}$$

Let α_c denote the anomaly theory from Section 3.2, and let $\sigma_{X,c}$ denote the theory from Section B.1.

Theorem 3.12. *An X-anomaly $F_c\colon \alpha_c \to 1_X$ of F (as in Definition 3.4) naturally defines:*

(i) A projective X-theory $\overline{F}\colon \mathbf{Bord}_d^X \to \mathbb{P}\Omega\mathcal{T}$ with projectivity α_c and underlying theory F (as in Definition 2.17).
(ii) $\widetilde{F}\colon \mathbf{Bord}_d^{\widetilde{X}} \to \Omega\mathcal{T}$ with underlying framed theory, written F.
(iii) Assuming Hypothesis Q, a $(\sigma_{X,c}, \rho_{X,c})$-module structure on F (Section B.2).

Proof. The fact that F_c determines (i) follows from Proposition 3.5. Similarly, Proposition 3.8 shows that F_c determines (ii). Finally, the fact that F_c determines (iii) follows from Proposition B.14 (which depends on Hypothesis Q). □

Theorem 3.13. *Consider the setup of Theorem 3.12, and assume that the cocycle c is trivializable after being pulled back along a map $f\colon Y \to X$. A trivialization of f^*c naturally defines a trivialization of the anomaly theory α_{f^*c} (as in (3.11)) which in turn defines:*

(i) A linearization of $\overline{F}$ (as in Proposition 2.18).
(ii) A Y-theory F_Y which factors $f^*\widetilde{F}$ (as in Question 2).
(iii) Assuming Hypothesis Q, a reduction (Definition B.24) from the $(\sigma_{X,c}, \rho_{X,c})$-module structure in Theorem 3.12 (iii) to a (σ_Y, ρ_Y)-module structure.

Furthermore, the absolute Y-theories obtained from (i), (ii), and (iii) agree with the one obtained from the trivialization of the anomaly.

Note that if one such trivialization exists, then the collection of trivializations forms a torsor over $H^d(X, \mathbf{k}^\times)$.

Proof. In terms of Question 2, a trivialization of $\widetilde{X}$ over Y (i.e. a trivialization of f^*c) explicitly defines a section $\mathbf{Bord}_d^Y \to \mathbf{Bord}_d^{f^*\widetilde{X}}$, which we can compose with $f^*\widetilde{F}$ to obtain the desired theory F_Y.

The theory α_{f^*c} (from (3.11)) is defined by the map f^*c, so a trivialization of f^*c automatically provides a trivialization $1_Y \xrightarrow{\sim} \alpha_{f^*c}$. This produces an automorphism of the trivial Y-theory $1_Y \xrightarrow{\sim} \alpha_{f^*c} \to 1_Y$, which is a theory on $\mathbf{Bord}_d^Y$ by Proposition A.9. This Y-theory is classified (via the Cobordism Hypothesis (A.6)) by the functor $Y \to \Omega\mathcal{T}$ given by the composition of the section of $f^*\widetilde{X}$ with the functor $\widetilde{X} \to \Omega\mathcal{T}^\sim$ classifying the anomalous theory:

$$Y \to f^*\widetilde{X} \to \Omega\mathcal{T}^\sim .$$

This agrees with the functor $Y \to \Omega\mathcal{T}^\sim$ classifying F_Y by Theorem 3.12. The agreement between the trivialization of the anomaly and the linearization (i) is Proposition 2.18.

A trivialization of f^*c determines a reduction to $\left(\sigma_Y^{d+1}, \rho_Y\right)$ by Proposition B.23. In particular, the boundary theory $1 \to \sigma_{\widetilde{X}}^{d+1}$ becomes reduced to a boundary theory $1 \to \sigma_Y^{d+1}$ which defines a Y-theory by Proposition B.14. This agrees with F_Y by the commuting square (B.7) in Remark B.5 associated to $X_1 = X$, $X_2 = \widetilde{X}$, and the morphism in (B.22). □

3.4. **A universal anomaly.** Let

$$F\colon \mathbf{Bord}_d^{\mathrm{fr}} \to \Omega\mathcal{T}$$

be a TQFT sending the point to an object $\mathcal{S}$ of $\Omega\mathcal{T}$. Let $\widetilde{X} = B\operatorname{Aut}_{\Omega\mathcal{T}}(\mathcal{S})$. Because we are assuming $\Omega^{d+1}\mathcal{T} \simeq \mathbf{k}$ (see the beginning of Section A.1), we know that $\mathbf{k}^\times$ maps to $\Omega^{d-1}\operatorname{Aut}(\mathcal{S})$, however it is only an equivalence when $\mathcal{S}$ satisfies a sort of simplicity/irreducibility.

Definition 3.14. We say an object $\mathcal{S}$ of $\Omega\mathcal{T}$ is a *Schur object* if $\Omega^{d-1}\operatorname{Aut}(\mathcal{S}) \cong \mathbf{k}^\times$.

If $\mathcal{S}$ is a Schur object, then the top level of the Postnikov tower provides a projectivity class c on the truncation to a $(d-1)$-type:

$$\begin{array}{c} B\operatorname{Aut}_{\Omega\mathcal{T}}(\mathcal{S}) \\ \downarrow{\scriptstyle \pi} \\ \pi_{\leq d-1} B\operatorname{Aut}_{\Omega\mathcal{T}}(\mathcal{S}) \xrightarrow{c} B^{d+1}\mathbf{k}^{\times} \ . \end{array} \tag{3.15}$$

The following follows from Theorems 3.12 and 3.13 by setting

$$\widetilde{X} = B\operatorname{Aut}_{\Omega\mathcal{T}}(\mathcal{S}) \ , \qquad X = \pi_{\leq d-1} B\operatorname{Aut}_{\Omega\mathcal{T}}(\mathcal{S}) \ , \text{ and } \qquad Y = BG \tag{3.16}$$

for some (possible higher) group G.

Corollary 3.17. *Let $\mathcal{S}$ be a Schur object (Definition 3.14) of an $(\infty, d+1)$-category $\mathcal{T}$ with duals which satisfies $\Omega^{d+1}\mathcal{T} = \mathbf{k}$.*

- *The class c from (3.15) uniquely determines the following.*
  - *(1) The gerbe $B\operatorname{Aut}_{\Omega\mathcal{T}}(\mathcal{S}) \to X$ of objects of $\Omega\mathcal{T}$ equivalent to $\mathcal{S}$.*
  - *(2) The anomaly theory α_c (as in Section 3.2).*
  - *(3) The abstract projectivity α_c (as in Definition 2.17).*
  - *(4) The quiche $(\sigma_{X,c}, \rho_{X,c})$ (as in Section B.2).*
- *The object $\mathcal{S}$ uniquely determines the following.*
  - *(i) An $\widetilde{X}$-theory $\widetilde{F}\colon \mathbf{Bord}_d^{B\operatorname{Aut}(\mathcal{S})} \to \mathcal{T}$.*
  - *(ii) An X-anomaly $F_c\colon \alpha_c \to 1$ (as in Definition 3.4).*
  - *(iii) A projective theory $\overline{F}$ with projectivity α_c (as in Definition 2.17).*
  - *(iv) Assuming Hypothesis Q, a $(\sigma_{X,c}, \rho_{X,c})$-module structure on F (as in Section B.2).*
- *Given $f\colon BG \to X$ for some (possibly higher) group G, a trivialization of f^*c determines the following.*
  - *(a) A theory F_{BG} which factors $\widetilde{F}$ (as in Question 2).*
  - *(b) A trivialization of the anomaly α_{f^*c} (as in (3.11)).*
  - *(c) A linearization of the projective theory $\overline{F}$ (as in Proposition 2.18).*
  - *(d) Assuming Hypothesis Q, a reduction to a (σ_{BG}, ρ_{BG})-structure on F (as in Definition B.24).*

  Furthermore, the trivialization in (b), the linearization in (c), and the reduction in (d) determine three BG-theories, which all agree with F_{BG}. If such a trivialization exists, then the trivializations of f^*c form a torsor over $H^d(BG, \mathbf{k}^{\times})$.

4. Fusion categories

4.1. **Preliminaries.** Fix an algebraically closed field $\mathbf{k}$ of characteristic zero. Let $\mathbf{Pr}$ be the symmetric monoidal 2-category of presentable $\mathbf{k}$-linear categories with colimit preserving functors (and natural transformations).[8]

As in [BJS21, §3], we will consider the "even higher" Morita category [JFS17] of $\mathbb{E}_2$-categories. This Morita category of braided monoidal categories was proposed in [Wal06, §9], [DSPS20], and [BZBJ18]. See Remark 2.12 for more details on the specific model of the Morita category used here.

The Morita 3-category of tensor categories, $\mathbf{Tens} = \mathbf{Alg}_1(\mathbf{Pr})$, consists of the following:

- the objects are tensor categories,
- the 1-morphisms are bimodule categories,
- the 2-morphisms are functors between bimodule categories, and
- the 3-morphisms are natural transformations.

[8] A category is presentable (sometimes called locally presentable) if it is accessible (generated under colimits by a small subcategory) and cocomplete (closed under small colimits).

We will focus on the subcategory **Fus** consisting of fusion categories, semisimple bimodule categories, compact-preserving cocontinuous bimodule functors, and natural transformations. It was shown in [BJS21] that this forms a subcategory.

Following [BJS21], we define "fusion" as follows. Given an $\mathbb{E}_1$-algebra object of **Pr**, there is an increasingly strict list of finiteness conditions one can insist upon:

- cp-rigid: every compact projective object is dualizable.
- compact-rigid: every compact object is dualizable.
- finite: compact rigid and the underlying category is finite.
- fusion: finite, semisimple,[9] with simple unit.

Remark 4.1. Recall the original notions of rigid, finite, and fusion abelian categories as in [EO04, ENO05, EGNO15]. As explained in [BJS21, BJSS21] this setting is related to the one defined above by an Ind-completion, in the following sense. The ambient category of abelian categories is not suitable for TQFT constructions, because it is not closed under the (Deligne) tensor product. Instead, as in [Shi20], we can work in the ambient category of finitely co-complete **k**-linear categories with right exact functors and natural equivalences, written **Rex**.

As is explained in [BZBJ18, §3], taking the Ind-completion defines an equivalence between **Rex** and the subcategory $\mathbf{Pr}_c \subset \mathbf{Pr}$ consisting of compactly generated **k**-linear presentable categories, compact functors, and natural equivalences.[10] As in [BZBJ18, §3], Ind and the functor taking compact objects of a presentable category define an equivalence of $(2,1)$-categories $\mathbf{Rex} \simeq \mathbf{Pr}_c$.

As it turns out, if an abelian category is rigid, then its Ind-completion is compact-rigid in the above sense [BJS21, Remark 1.6]. Similarly, if an abelian category is finite (in the usual sense [EO04]) then its Ind-completion is finite (compact-rigid and finite underlying category), since the compact objects of the Ind-completion are the objects of the original category [BJS21, Remark 2.11].

Check: BJS Remark 2.11 is the $\mathbf{Rex}/\mathbf{Pr}_c$ comparison; it does not state the "finite" claim verbatim.

The upshot of this discussion is that the Ind-completion of a fusion category in the traditional sense lands in the category $\mathbf{Fus} \subset \mathbf{Alg}_1(\mathbf{Pr})$ as it is defined above. Note that the Deligne-Kelly tensor product coincides with the Deligne tensor product of finite abelian categories (and commutes with taking ind-completions), see the proof of [BJS21, Definition-Proposition 4.8]. Therefore Ind identifies the Morita 3-category of fusion categories in the abelian sense with $\mathbf{Fus} \subset \mathbf{Alg}_1(\mathbf{Pr})$, and any statement about the former which is invariant under equivalence holds for the latter. In particular, we will use various facts from [DSPS20, ENO10] (which are in the abelian setting) throughout.

4.1.1. *Braided fusion categories.* Similarly, braided tensor categories form a 4-category $\mathbf{BrTens} = \mathbf{Alg}_2(\mathbf{Pr})$:

- the objects are braided tensor categories,
- the 1-morphisms are associative algebra objects in the category of bimodule categories,
- the 2-morphisms are bimodule categories between bimodule categories,
- the 3-morphisms are functors between bimodule categories, and
- the 4-morphisms are natural transformations.

The subcategory **BrFus** consists of braided fusion categories, fusion categories equipped with central structures, finite semisimple bimodule categories, compact-preserving cocontinuous bimodule functors, and bimodule natural transformations. It was shown in [BJS21] that this forms a subcategory.

Braided fusion categories also form a 2-category:

- the objects are braided fusion categories,
- the 1-morphisms are functors preserving the braided structure, and

[9] By semisimple, we mean [BJS21, Definition 2.27]: Every object is a (possibly infinite) direct sum of simple objects.

[10] A functor is compact if it preserves compact objects.

- the 2-morphisms are natural transformations.

The 2-groupoid given by the invertible part of this 2-category is written **EqBr**, as in [ENO10].

4.2. **Higher groupoids attached to a braided fusion category.** Let $\mathcal{A}$ be a braided fusion category. The braided equivalences of $\mathcal{A}$ form a 2-group $\mathrm{Aut}_{\mathbf{EqBr}}(\mathcal{A})$. From any 2-group, we can construct its classifying space $B\,\mathrm{Aut}_{\mathbf{EqBr}}(\mathcal{A})$, which is a 2-type. I.e. it has two nontrivial homotopy groups.

To $\mathcal{A}$ we can also attach the 2-category of $\mathcal{A}$-module categories. The braiding defines an embedding from the 2-category of $\mathcal{A}$-modules to the 2-category of $\mathcal{A}$-bimodules. Therefore the category of $\mathcal{A}$-modules inherits a monoidal structure from the natural one on the category of $\mathcal{A}$-bimodules.

Definition 4.2. The invertible $\mathcal{A}$-modules, written $\mathrm{Pic}(\mathcal{A})$, comprise the *Picard 3-group of* $\mathcal{A}$.

From any 3-group, we can construct its classifying space $B\,\mathrm{Pic}(\mathcal{A})$, which is a 3-type. I.e. it has three nontrivial homotopy groups.

Remark 4.3. Some authors write the full 3-group/type as $B\underline{\underline{\mathrm{Pic}}}(\mathcal{A})$, and write the truncation to a 2-type as $B\underline{\mathrm{Pic}}(\mathcal{A})$ and to a 1-type as $B\,\mathrm{Pic}(\mathcal{A})$.

Instead, we write $B\,\mathrm{Pic}(\mathcal{A})$ for the full 3-group, and write $\tau_{\leq 2}B\,\mathrm{Pic}(\mathcal{A})$ and $\tau_{\leq 1}B\,\mathrm{Pic}(\mathcal{A})$ for the truncations.

Theorem ([ENO10, Theorem 5.2])**.** *For a nondegenerate braided fusion category $\mathcal{A}$ we have an equivalence:*

$$\pi_{\leq 2}B\,\mathrm{Pic}(\mathcal{A}) \simeq B\,\mathrm{Aut}_{\mathbf{EqBr}}(\mathcal{A}) \ . \tag{4.4}$$

The homotopy groups of $B\,\mathrm{Pic}(\mathcal{A})$, e.g. from [ENO10, Proposition 7.5] are as follows: π_1 is given by the ordinary group of isomorphism classes of braided equivalences, and π_2 is given by the group of tensor isomorphisms of the identity functor on $\mathcal{A}$. The top homotopy group, which is not involved in (4.4), is $\pi_3 B\,\mathrm{Pic}(\mathcal{A}) = \mathbf{k}^\times$.

4.2.1. *Braided fusion categories which are Drinfeld centers.* Let $\mathcal{C}$ be a fusion category. To this we can attach the monoidal 2-category $\mathrm{End}_{\mathbf{Fus}}(\mathcal{C})$.

Definition 4.5. The *Brauer-Picard 3-group* of $\mathcal{C}$ is $\mathrm{Aut}_{\mathbf{Fus}}(\mathcal{C})$.

Recall the Drinfeld center of a monoidal category $\mathcal{C}$ is a braided monoidal category, written $\mathcal{Z}(\mathcal{C})$.

Theorem ([EO04])**.** *There is an equivalence of 2-categories:*

$$\mathrm{End}_{\mathbf{Fus}}(\mathcal{C}) \xrightarrow{\sim} \mathcal{Z}(\mathcal{C})\text{-}\mathbf{mod} \ .$$

Passing to the invertible part, we obtain:

$$\mathrm{Aut}_{\mathbf{Fus}}(\mathcal{C}) \simeq \mathrm{Pic}(\mathcal{Z}(\mathcal{C})) \ . \tag{4.6}$$

Remark 4.7. In [ENO11], it is shown that two fusion categories are Morita equivalent if and only if their Drinfeld centers are braided equivalent. This was strengthened in [ENO10]: they prove that there is a fully-faithful embedding of groupoids $\mathbf{EqBr} \to \mathbf{Fus}^\times$. Indeed, combining (4.4) and (4.6) we obtain

$$\pi_{\leq 2}B\,\mathrm{Aut}_{\mathbf{Fus}}(\mathcal{C}) \simeq B\,\mathrm{Aut}_{\mathbf{EqBr}}(\mathcal{Z}(\mathcal{C})) \ . \tag{4.8}$$

So the center determines the Morita class of the fusion category itself. However, since the equivalence (4.4) is with the truncated part of $\mathrm{Aut}_{\mathbf{Fus}}(\mathcal{C})$, there is a subtle lack of "coherence", captured by an obstruction class originally studied in [ENO10] and surveyed in Section 4.4.3.

Related results were shown in [KZ18] and [KK12, §5].

In summary: given a braided monoidal category $\mathcal{A}$, we have a 3-type $B\operatorname{Pic}(\mathcal{A})$ attached to it. If it happens to be the case that $\mathcal{A} = \mathcal{Z}(\mathcal{C})$ for some fusion category $\mathcal{C}$, then this 3-type is the classifying space of the Brauer-Picard 3-group of $\mathcal{C}$. All together we have:

$$\begin{array}{ccc}
\operatorname{End}_{\mathbf{Fus}}(\mathcal{C}) & \xrightarrow[\text{[EO04]}]{\sim} & \mathcal{Z}(\mathcal{C})\text{-}\mathbf{mod} \\
\\
B\operatorname{Aut}_{\mathbf{Fus}}(\mathcal{C}) & \xrightarrow{\sim} & B\operatorname{Pic}(\mathcal{Z}(\mathcal{C})) \\
\downarrow & & \downarrow \\
\pi_{\leq 2} B\operatorname{Aut}_{\mathbf{Fus}}(\mathcal{C}) & \xrightarrow{\sim} & \pi_{\leq 2} B\operatorname{Pic}(\mathcal{Z}(\mathcal{C})) \\
& {\scriptstyle\sim}\searrow \quad \swarrow{\scriptstyle\sim}\ \text{[ENO10]} & \\
& B\operatorname{Aut}_{\mathbf{EqBr}}(\mathcal{Z}(\mathcal{C})) &
\end{array}$$

4.3. **Pointed braided fusion categories.** A fusion category is *pointed* if all simple objects are invertible.

Example 4.9. Let L be a finite group. Consider the category of L-graded vector spaces, $\mathbf{Vect}[L]$, with convolution, i.e. for $a, b, c \in L$:

$$(W * W')_a = \bigoplus_{a=bc} W_b \otimes W_c \ .$$

This category has simple objects $\mathbf{k}_a$ for $a \in L$, and is therefore pointed. Given a 3-cocycle τ on L, we can define a variant of this category $\mathbf{Vect}[L]^\tau$, in which the associators are multiplied by τ [EGNO15, Example 2.3.8]. This category is still pointed.

All pointed fusion categories are of the form $\mathbf{Vect}[L]^\tau$ [EGNO15]. (Note however that the twisted version may have no fiber functor. This is a higher analogue of the fact that not all algebras have an augmentation map.)

4.3.1. *Metric groups.* Metric groups [DGNO10, ENO10] play an important role in the theory of pointed braided fusion categories. Let $\mathbf{k}$ be an algebraically closed field of characteristic zero, and let A be a finite abelian group. Write $A^* \coloneqq \operatorname{Hom}(A, \mathbf{k}^\times)$ for the character dual of A.

Definition 4.10. A biadditive map

$$\langle \cdot, \cdot \rangle : A \times A \to \mathbf{k}^\times$$

is a *symmetric bicharacter* if for all $a, b \in A$ we have $\langle a, b \rangle = \langle b, a \rangle$. By acting on the first argument, such a bicharacter $\langle \cdot, \cdot \rangle$ defines a homomorphism $A \to A^*$. If this is an isomorphism, then we say that $\langle \cdot, \cdot \rangle$ is *nondegenerate*.

Definition 4.11. A function

$$q \colon A \to \mathbf{k}^\times$$

is a *quadratic form* on A if

(a) for all $a \in A$ we have $q(a) = q(-a)$, and
(b) the following expression defines a symmetric bicharacter:

$$\langle a, b \rangle_q \coloneqq \frac{q(a+b)}{q(a)\, q(b)} \ .$$

We say q is *nondegenerate* if the bicharacter $\langle \cdot, \cdot \rangle_q$ is nondegenerate.

Remark 4.12. If $|A|$ is odd, then sending q to $\langle \cdot, \cdot \rangle_q$ defines a bijection between quadratic forms and bicharacters.

Definition 4.13. Any pair (A, q) is a *pre-metric group*. A pair (A, q) is called a *metric group* if q is nondegenerate.

Definition 4.14. The *orthogonal group* of a metric group (A, q) is:

$$\mathrm{O}(A, q) \coloneqq \{f \in \mathrm{Hom}(A, A) \,|\, q \circ f = q\} \ .$$

Definition 4.15. Let (A, q) be a metric group. Define the *determinant*

$$\det\colon \mathrm{O}(A, q) \to \mathbb{Q}^\times{}_{>0} / \left(\mathbb{Q}^{\times 2}{}_{>0}\right)$$

by sending $g \in \mathrm{O}(A, q)$ to the image of $|(g-1)A| \in \mathbb{N}$.

The *special orthogonal group* is

$$\mathrm{SO}(A, q) \coloneqq \ker(\det) \ . \tag{4.16}$$

4.3.2. *Classification of pointed braided fusion categories.* It is shown in [JS93] that pointed braided fusion categories are classified by finite abelian groups A (the group of simple objects) equipped with a quadratic form $q\colon A \to \mathbf{k}^\times$ which is not necessarily nondegenerate. See also [EGNO15, §8.4].

Example 4.17. If L is any finite abelian group, then the finite abelian group $L \oplus L^*$ has a non-degenerate quadratic form given by evaluation, so $(L \oplus L^*, \mathrm{ev})$ is a metric group. The symmetric bicharacter corresponding to evaluation is:

$$\langle (\ell, \chi), (\ell', \chi') \rangle = \chi(\ell')\chi'(\ell) \ .$$

Given an abelian group A, it is shown in [EL54, Theorem 26.1] that $H^4\left(B^2 A, \mathbf{k}^\times\right)$ is isomorphic to the group of quadratic forms $A \to \mathbf{k}^\times$.

Eilenberg and Mac Lane defined an explicit chain complex which computes this cohomology [ML52, EML50a, EML50b, EML51]. They call this abelian group cohomology, and write it as H^*_{ab}:

$$H^n_{\mathrm{ab}}(A, B) = H^{n+1}(K(A, 2), B)$$

for any two abelian groups A and B. I.e. we have

$$H^3_{\mathrm{ab}}\left(A, \mathbf{k}^\times\right) = H^4\left(B^2 A, \mathbf{k}^\times\right) \cong \mathrm{Quad}\left(A, \mathbf{k}^\times\right) \ . \tag{4.18}$$

Cocycle representatives in $H^3_{\mathrm{ab}}(A, \mathbf{k}^\times)$ are pairs (τ, b) where τ is an ordinary group cocycle on A, and b is a map $A^2 \to \mathbf{k}^\times$ satisfying the two hexagon equations with τ [JS93] (see [EGNO15] for explicit formulas). Explicitly, the identification (4.18) sends a pair (τ, b) to the quadratic form defined by $q(a) \coloneqq b(a, a)$. Given such a pair (τ, b), we obtain a not necessarily symmetric braided structure on the monoidal category of A-graded vector spaces with associator twisted by τ. We can specify this braiding on simple objects:

$$\mathbf{k}_a * \mathbf{k}_b = \mathbf{k}_{a+b} \xrightarrow[\sim]{b(a,b)\cdot \mathrm{id}_{\mathbf{k}_{a+b}}} \mathbf{k}_{a+b} = \mathbf{k}_b * \mathbf{k}_a \ . \tag{4.19}$$

Add the precise section of EGNO containing the hexagon equations for abelian 3-cocycles.

Example 4.20. This is a continuation of Example 4.17. Recall the pointed braided fusion category $\mathcal{Z}(\mathbf{Vect}[L])$ was classified by the finite abelian group $A = L \oplus L^*$ with quadratic form $q = \mathrm{ev}$.

A degree-3 cocycle in abelian group cohomology is a pair (τ, b) where τ is an ordinary group 3-cocycle on A and $b\colon A^2 \to \mathbf{k}^\times$ (satisfying some conditions).

The abelian group 3-cocycle classifying $\mathcal{Z}(\mathbf{Vect}[L])$ has trivial τ, and $b = b_0$ is a map from A^2 to $\mathbf{k}^\times$ such that it agrees with q when restricted to the diagonal, i.e. for all $a \in A$:

$$b_0(a, a) = \mathrm{ev}(a) \ .$$

For example, b_0 might be taken to be:

$$b_0((\ell_1, \chi_1), (\ell_2, \chi_2)) = \chi_1(\ell_2) \ .$$

Other choices for b_0 differ from this by an abelian group coboundary in degree 3.

4.3.3. *Higher groupoids in the pointed case.* When the (braided) fusion categories in question are pointed, the groupoids introduced in Section 4.2 have a more concrete description. Let (A, q) be a metric group, and let $\mathcal{A} = \mathbf{Vect}[A]$ be the associated braided category of A-graded vector spaces. The Picard 3-type of $\mathcal{A}$ has the following homotopy groups:

$$\pi_1 = \mathrm{O}(A, q) \qquad \pi_2 = A \qquad \pi_3 = \mathbf{k}^\times \ . \tag{4.21}$$

Remark 4.22. Following [ENO10, Remark 10.7], these homotopy groups appear as follows. First of all, we know that $\pi_2 = A$ and $\pi_3 = \mathbf{k}^\times$, since this is the underlying groupoid of $\mathcal{A}$ itself (delooped twice).

Now we identify π_1. The Whitehead half-square (see [ENO10, §7.3]) is a homotopy invariant, which is a map $\pi_2 \to \pi_3$. As it turns out, it is given by the self-braiding on $\mathcal{A}$, which is q in this case. The action of π_1 on π_2 by conjugation must preserve this invariant, and hence this defines a map $\pi_1 \to \mathrm{O}(A, q)$, which turns out to be an equivalence.

This conjugation action is a shadow of a more general way to understand these groupoids: The invertible part of any monoidal 2-category acts on the endomorphisms of the identity object by conjugation. As is explained in [ENO10, Remark 5.4], by setting the monoidal 2-category to be $\mathrm{End}_{\mathbf{Fus}}(\mathcal{C})$, we obtain a map:

$$\mathrm{Aut}_{\mathbf{Fus}}(\mathcal{C}) \to \mathrm{Aut}_{\mathbf{EqBr}}(\mathcal{Z}(\mathcal{C})) \ .$$

When we restrict this to the truncation of $\mathrm{Aut}_{\mathbf{Fus}}(\mathcal{C})$ to a 1-category (i.e. 2-group), we obtain the equivalence (4.8). When we set the 2-category to be the category of modules over $\mathcal{Z}(\mathcal{C})$, we obtain the equivalence (4.4).

4.3.4. *Lagrangian subgroups.*

Definition 4.23. A subgroup $L \subset A$ of a metric group (A, q) is *isotropic* if $q(\ell) = 1$ for all $\ell \in L$. The subgroup is *Lagrangian* if it is isotropic and $|L|^2 = |A|$.

Example 4.24. If $A = L \oplus L^*$ and $q = \mathrm{ev}$, then $L \oplus \{0\}$ and $\{0\} \oplus L^*$ are both Lagrangian subgroups.

Let L be a finite abelian group. The Drinfeld center of the fusion category $(\mathbf{Vect}[L], *)$ is:

$$\mathcal{Z}(\mathbf{Vect}[L], *) \cong \mathbf{Vect}[L \oplus L^*] \ . \tag{4.25}$$

Indeed, a simple object of the center consists of a simple object $\mathbf{k}_\ell$, $\ell \in L$, together with a half-braiding, which is given by a character $\chi \in L^*$. The tensor structure is convolution, and the braiding is induced by evaluation, as in Example 4.17.

Recall from Section 4.3.2, that pointed braided fusion categories are classified by finite abelian groups equipped with a quadratic form. In this case, the finite abelian group associated to $\mathcal{Z}(\mathbf{Vect}[L])$ is $L \oplus L^*$, and the quadratic form is $\mathrm{ev}\colon L \oplus L^* \to \mathbf{k}^\times$. Lagrangians in the metric group give rise to fusion categories $\mathcal{C}$ such that $\mathcal{A} \cong \mathcal{Z}(\mathcal{C})$.

4.3.5. *Pointed Drinfeld centers.* Let $\mathcal{C} = \mathbf{Vect}[L]^\tau$ for L a finite group and τ a 3-cocycle on L. Following [MN18, §4], for any ℓ in the center of L, we can define:

$$T_\ell(x, y) = \frac{\tau(\ell, x, y)\,\tau(x, y, \ell)}{\tau(x, \ell, y)} \ . \tag{4.26}$$

As is shown in [MN18, §4.1], T_ℓ is a 2-cocycle on the group L, and $T_{(-)}$ defines a group homomorphism from the center of L to $H^2(L, \mathbf{k}^\times)$. Then, as in [MN18, Corollary 4.3], $\mathcal{Z}(\mathcal{C})$ is pointed if and only if L is abelian and $T_{(-)}$ is the trivial group homomorphism.

Example 4.27. When L is a vector space over $\mathbb{F}_p$, by [MN18, Corollary 5.3] $\mathcal{Z}(\mathcal{C})$ is pointed if and only if τ has trivial alternating component. (Recall $H^3(L, \mathbf{k}^\times)$ decomposes as a direct sum of the alternating factor $\wedge^3 L^*$ and the symmetric factor $\mathrm{Sym}^2 L^*$.)

4.3.6. *Polarizability.*

Definition 4.28. A pointed braided fusion category is *polarizable* if it is the Drinfeld center of another fusion category $\mathcal{C}$.

The usual classification of pointed braided fusion categories (Section 4.3.2) restricts to a classification of polarizable pointed braided fusion categories as follows.

Proposition 4.29. *A pointed braided fusion category classified by (A, q) is polarizable if and only if q is nondegenerate and A contains a Lagrangian subgroup L* [DGNO10, §4.4]. *In this case $\mathcal{C}$ can be taken to be pointed, i.e. of the form $\mathbf{Vect}\,[L^*]^\tau$, and A is an extension:*

$$0 \to L \to A \to L^* \to 0$$

where the second map is induced by $\langle \cdot, \cdot \rangle_q$.

We call such a triple (A, q, L) a *polarized metric group.*

Example 4.30. This extension need not split. For example, let τ be a nontrivial 3-cocycle on $\mathbb{Z}/3$. Since $H^2(\mathbb{Z}/3, \mathbf{k}^\times)$ vanishes, $\mathcal{Z}(\mathbf{Vect}\,[\mathbb{Z}/3]^\tau)$ is pointed by Section 4.3.5, but its group of simple objects is $\mathbb{Z}/9$.

4.4. **Obstruction theory.** We give an executive summary of the obstruction theory developed in [ENO10].

4.4.1. *Postnikov and Whitehead towers.* Recall the definition of the Postnikov and Whitehead towers of a space, and the associated k-invariants. See [Pos55, Whi78], or [MP12, Chapter 3] for references.

Using the identifications (4.6) and (4.4), the 3-type $B\,\mathrm{Aut}_{\mathbf{Fus}}(\mathcal{C}) \simeq B\,\mathrm{Pic}(\mathcal{Z}(\mathcal{C}))$ has Postnikov and Whitehead towers:

$$\begin{array}{ccccccc}
* & \longrightarrow & B^3\pi_3 & \longrightarrow & B^2\mathcal{Z}(\mathcal{C})^\times & \longrightarrow & B\,\mathrm{Aut}_{\mathbf{Fus}}(\mathcal{C}) \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & * & \longrightarrow & B^2\pi_2 & \longrightarrow & B\,\mathrm{Aut}_{\mathbf{EqBr}}(\mathcal{Z}(\mathcal{C})) \\
 & & & & \downarrow & & \downarrow \\
 & & & & * & \longrightarrow & B\pi_1 \\
 & & & & & & \downarrow \\
 & & & & & & *
\end{array} \tag{4.31}$$

Recall the homotopy groups π_i from (4.21) and Remark 4.22. The 1-connected cover of $B\,\mathrm{Aut}_{\mathbf{Fus}}(\mathcal{C}) \simeq B\,\mathrm{Pic}(\mathcal{Z}(\mathcal{C}))$ is the classifying space of the connected component of the identity in $\mathrm{Pic}(\mathcal{Z}(\mathcal{C}))$, i.e. of the regular module $\mathcal{Z}(\mathcal{C})_{\mathcal{Z}(\mathcal{C})}$. The automorphisms of the regular module are the invertible objects of $\mathrm{End}_{\mathcal{Z}(\mathcal{C})}(\mathcal{Z}(\mathcal{C})) \simeq \mathcal{Z}(\mathcal{C})$, so this cover is the underlying groupoid of $\mathcal{Z}(\mathcal{C})^\times$ delooped twice, $B^2\mathcal{Z}(\mathcal{C})^\times$ (as in Remark 4.22). See Remark 5.2 for more on the space $B^2\mathcal{Z}(\mathcal{C})^\times$.

4.4.2. *Degree three obstruction.* Let $\mathcal{A}$ be a nondegenerate braided fusion category, e.g. $\mathcal{A} = \mathcal{Z}(\mathcal{C})$.

Question 3. Given a group G and a morphism of groups $f\colon G \to \pi_1$, what is the obstruction to performing the following lift?

$$\begin{array}{ccc}
 & & \mathrm{Aut}_{\mathbf{EqBr}}(\mathcal{A}) \\
 & \overset{\exists}{\nearrow} & \downarrow \\
G & \xrightarrow{f} & \pi_1
\end{array}$$

This question is equivalent to asking if the k-invariant $k_2 \in H^3(B\pi_1, \pi_2)$ of the 2-type $B\operatorname{Aut}_{\mathbf{EqBr}}(\mathcal{A})$ pulls back to something trivializable on G. I.e. the pullback

$$O_3(f) \coloneqq (Bf)^* k_2 \in H^3(G, \pi_2) \tag{4.32}$$

is the obstruction to lifting:

$$\begin{array}{ccc} & & B\operatorname{Aut}_{\mathbf{EqBr}}(\mathcal{A}) \\ & \nearrow & \downarrow \\ BG & \xrightarrow[Bf]{} & B\pi_1 = \pi_{\leq 1} B\operatorname{Aut}_{\mathbf{EqBr}}(\mathcal{A}) \end{array}$$

Remark 4.33. At the level of classifying spaces, (4.4) tells us that the truncation of $B\operatorname{Pic}(\mathcal{A})$ to a 2-type is identified with $B\operatorname{Aut}_{\mathbf{EqBr}}(\mathcal{A})$. Therefore O_3 is the same as the obstruction to lifting:

$$\begin{array}{ccccc} & & \pi_{\leq 2} B\operatorname{Pic}(\mathcal{A}) & & \\ & \nearrow & & \searrow & \\ BG & \xrightarrow{Bf} & B\pi_1 & \simeq & \pi_{\leq 1} B\operatorname{Pic}(\mathcal{A}) \end{array}$$

Remark 4.34. When $\mathcal{A} = \mathcal{Z}(\mathcal{C})$ for some fusion category $\mathcal{C}$, (4.6) allows us to describe this obstruction in terms of the Brauer-Picard 3-type $B\operatorname{Aut}_{\mathbf{Fus}}(\mathcal{C})$. In particular, it is the same as the obstruction to lifting:

$$\begin{array}{ccccc} & & \pi_{\leq 2} B\operatorname{Aut}_{\mathbf{Fus}}(\mathcal{C}) & & \\ & \nearrow & & \searrow & \\ BG & \xrightarrow{Bf} & B\pi_1 & \simeq & \pi_{\leq 1} B\operatorname{Aut}_{\mathbf{Fus}}(\mathcal{C}) \end{array}$$

4.4.3. *Degree four obstruction in the split case.* Assume that the obstruction $O_3(f)$ is trivializable, and let s be the section of $B\operatorname{Aut}_{\mathbf{EqBr}}(\mathcal{Z}(\mathcal{C}))$ over $B\pi_1$ corresponding to a fixed trivialization of $O_3(f)$. Recall the trivializations of $O_3(f)$ form a torsor over $H^2(BG, \pi_2)$. By Remark 4.33, the upshot of the existence of s is that we have an associated map $Bf_s : BG \to \pi_{\leq 2} B\operatorname{Pic}(\mathcal{A})$.

Question 4. What is the obstruction to performing the following lift?

$$\begin{array}{ccc} & & B\operatorname{Pic}(\mathcal{A}) \\ & \nearrow & \downarrow \\ BG & \xrightarrow{Bf_s} & \pi_{\leq 2} B\operatorname{Pic}(\mathcal{A}) \end{array}$$

Just as before, the obstruction comes from the k-invariant. Now the relevant k-invariant is

$$k_3 \in H^4(\pi_{\leq 2} B\operatorname{Pic}(\mathcal{A}), \pi_3) \ . \tag{4.35}$$

The pullback

$$O_4(f, s) \coloneqq (Bf_s)^* k_3 \in H^4(BG, \pi_3)$$

is the obstruction to lifting.

Remark 4.36. Again, if $\mathcal{A} \cong \mathcal{Z}(\mathcal{C})$ for some fusion category $\mathcal{C}$ then, by Remark 4.34, we have an associated map $Bf_s : BG \to \pi_{\leq 2} B\operatorname{Aut}_{\mathbf{Fus}}(\mathcal{C})$. By (4.6), the above question is equivalent to the following.

Question 5. What is the obstruction to performing the following lift?

$$\begin{array}{ccc} & & B\operatorname{Aut}_{\mathbf{Fus}}(\mathcal{C}) \\ & \nearrow & \downarrow \\ BG & \xrightarrow{Bf_s} & \pi_{\leq 2} B\operatorname{Aut}_{\mathbf{Fus}}(\mathcal{C}) \end{array}$$

The obstruction to lifting this is still $O_4(f,s) \in H^4(BG, \pi_3)$.

As it turns out, e.g. from [ENO10, Proposition 7.2, 7.3]:

$$\pi_3\left(B\operatorname{Aut}_{\mathbf{Fus}}(\mathcal{C})\right) \cong \mathbf{k}^\times \cong \pi_3\left(B\operatorname{Pic}(\mathcal{A})\right) \ .$$

Therefore the obstruction is classified by familiar $\mathbf{k}^\times$-cohomology:

$$O_4(f,s) \in H^4\left(BG, \mathbf{k}^\times\right) \ . \tag{4.37}$$

4.4.4. *Full degree four obstruction.* In Section 4.4.3, we assume that the map into $\pi_{\leq 2} B\operatorname{Pic}(\mathcal{A}) \simeq B\operatorname{Aut}_{\mathbf{EqBr}}(\mathcal{A})$ is given by a trivialization of $O_3(f)$ for f some map from a group to π_1. There is a slightly more general obstruction one can consider.

Question 6. Given a 2-group G_2, and a map

$$f_2 \colon BG_2 \to \pi_{\leq 2} B\operatorname{Pic}(\mathcal{A}) \simeq B\operatorname{Aut}_{\mathbf{EqBr}}(\mathcal{A}) \ ,$$

what is the obstruction to performing the following lift?

$$\begin{array}{ccc} & & B\operatorname{Pic}(\mathcal{A}) \\ & \overset{}{\nearrow} & \downarrow \\ BG_2 & \xrightarrow{f_2} & \pi_{\leq 2} B\operatorname{Pic}(\mathcal{A}) \end{array}$$

This is equivalent to asking if the k-invariant k_3 from (4.35) pulls back to something trivializable on G_2.

Remark 4.38. As in the preceding subsections, this is equivalent to the analogous lifting question for $B\operatorname{Aut}_{\mathbf{Fus}}(\mathcal{C})$, by (4.6). Furthermore, a combination of (4.4) and (4.6) tell us that we can consider the universal version of this question: $G_2 = \operatorname{Aut}_{\mathbf{EqBr}}(\mathcal{Z}(\mathcal{C}))$. I.e.

$$k_3 \colon B\operatorname{Aut}_{\mathbf{EqBr}}(\mathcal{Z}(\mathcal{C})) \to B^4\mathbf{k}^\times \tag{4.39}$$

is the obstruction to lifting

$$\begin{array}{ccc} & & B\operatorname{Aut}_{\mathbf{Fus}}(\mathcal{C}) \\ & \overset{}{\nearrow} & \downarrow \\ B\operatorname{Aut}_{\mathbf{EqBr}}(\mathcal{Z}(\mathcal{C})) & \xrightarrow{\sim} & \pi_{\leq 2} B\operatorname{Aut}_{\mathbf{Fus}}(\mathcal{C}) \end{array}$$

4.4.5. *Obstruction theory in the pointed case.* Recall the homotopy groups in (4.21). If we take $G = \mathrm{O}(A,q)$, we get a universal obstruction class:

$$O_3 \in H^3\left(B\,\mathrm{O}(A,q), A\right) \ .$$

Let $|A|$ be odd. Then, as discussed in [EG18, §6] (and [CGPW16, §5]), the obstruction O_3 vanishes, and in fact has a canonical splitting:

$$s \colon B\,\mathrm{O}(A,q) \to \pi_{\leq 2} B\operatorname{Pic}(\mathcal{A}) \ .$$

Therefore we have a well-defined universal version of the obstruction class in (4.37):

$$O_4(A,q) \in H^4\left(B\,\mathrm{O}(A,q), \mathbf{k}^\times\right) \ . \tag{4.40}$$

Remark 4.41. In other words, $s^* B\operatorname{Aut}_{\mathbf{Fus}}(\mathcal{C})$ defines a $B^3\mathbf{k}^\times$-bundle over $B\,\mathrm{O}(A,q)$. At the level of 3-groups, this is actually a *central* extension of $\mathrm{O}(A,q)$ by $B^2\mathbf{k}^\times$ since π_1 acts on $\pi_3 = \mathbf{k}^\times$ by conjugation, and conjugating a scalar automorphism by a $\mathbf{k}$-linear equivalence returns the same scalar.

Example 4.42. Let $G = \mathbb{Z}/2$. Then $H^2(BG, \mathbf{k}^\times) = 0$ so the class O_4 necessarily vanishes, and so the anomaly vanishes. There are two nonequivalent trivializations classified by $H^3(BG, \mathbf{k}^\times) = \mathbb{Z}/2$.

This is used in [ENO10, Example 9.4] to reproduce the classification in [TY98] of $\mathbb{Z}/2$-graded fusion categories.

Remark 4.43. Recall that the trivializations of $O_3(f)$ form a torsor over $H^2(BG, \pi_2)$. I.e. given two trivializations s and s' of $O_3(f)$, there exists $L \in H^2(BG, \pi_2)$ such that $s' = Ls$. By [ENO10, Proposition 8.15] we have that

$$O_4(f, s')/O_4(f, s) = \mathrm{PW}(L) \ ,$$

where PW denotes the *Pontrjagin-Whitehead quadratic function* from [ENO10, §8.7].

Also see [Jen11, Proposition 7.3] and [CGPW16, Proposition 8] for a concrete formula for the Pontrjagin-Whitehead quadratic function. It is shown to vanish in some examples in [GJ19].

4.4.6. *Vanishing of the obstruction over a finite field.* Let p be an odd prime, and let $A = V$ be a $2n$-dimensional vector space over the finite field $\mathbb{F}_p$. Let q_{split} be the quadratic form on V of signature (n, n).

Observation 4.44. *The orthogonal group* $\mathrm{O}(V, q_{\mathrm{split}})$ *is the split orthogonal group over* $\mathbb{F}_p$.

Note that, by (4.25), if L is a vector space over $\mathbb{F}_p$ of dimension n, then

$$\mathcal{Z}(\mathbf{Vect}[L], *) \simeq (\mathbf{Vect}[V], *, \beta_q) \ .$$

The obstruction $O_4(A, q_{\mathrm{split}})$ vanishes in this case by [EG18, Theorem 6.1].

Recall the analogy between this categorical representation of $\mathrm{O}(V, q)$ and the Weil representation of the metaplectic group in Remark 4.58. This vanishing can be thought as an analogue of the fact that the Weil representation splits over a finite field [GH09, GH11].

4.4.7. *Braided categories with a prescribed obstruction class.*

Proposition 4.45. *For any finite group* G *and group* 4*-cocycle* π *there exists a nondegenerate braided fusion category* $\mathcal{B}$ *and monoidal functor*

$$\rho \colon G \to \mathrm{Aut}_{\mathbf{EqBr}}(\mathcal{B})$$

such that $[\pi] = (B\rho)^* k_3$*, with* k_3 *as in* (4.35).

Proof. Recall that every fusion 2-category is Morita-equivalent to a connected fusion 2-category [Déc24, Theorem 4.2.2]. Therefore the fusion 2-category $2\mathbf{Vect}[G]^\pi$ is Morita equivalent to the 2-category of module categories over some braided fusion category $\mathcal{B}$.

In fact, we obtain more: As a special case of the classification of fusion 2-categories [DHJF+24], the connected ones are classified by a nondegenerate braided fusion category $\mathcal{B}$ equipped with a monoidal functor $\rho \colon G \to \mathrm{Aut}_{\mathbf{EqBr}}(\mathcal{B})$. The fact that $(B\rho)^* k_3 = [\pi]$ follows from the reconstruction of a connected fusion 2-category from $\mathcal{B}$ and ρ, as in [DHJF+24, §4.4]. □

4.5. **Extensions of the finite orthogonal group.** Throughout this section we assume $|A|$ is odd, as in Section 4.4.5. Recall, from Remark 4.41, that we can interpret $O_4 = O_4(A, q)$ (from Section 4.4.5) as encoding a central extension of $B\,\mathrm{O}(A, q)$ by $B^3\mathbf{k}^\times$. In particular, $\mathrm{hofib}(O_4)$ is a $B^3\mathbf{k}^\times$-bundle over $B\,\mathrm{O}(A, q)$:

$$\begin{array}{ccc} B^3\mathbf{k}^\times & \longrightarrow & \mathrm{hofib}(O_4) \\ \downarrow & & \downarrow \\ * & \longrightarrow & B\,\mathrm{O}(A, q) \end{array}$$

Definition 4.46. Let $3\mathrm{Lip}(A, q)$ denote the 3-group $\Omega\,\mathrm{hofib}(O_4)$, i.e. the 3-group with classifying space $\mathrm{hofib}(O_4)$.

Remark 4.47. The name comes from the analogy in Section 4.7. Namely, $\mathrm{3Lip}(A,q)$ is a 3-group analogue of the Lipschitz group (a.k.a. Clifford group) of a quadratic vector space.

Let $l = |L|$. The order of the class $O_4(A,q)$ is shown in [ENO10, Theorem 8.16] to divide l^4. In other words, it is the image of some class $c \in H^4(B\,\mathrm{O}(A,q), \mu_{l^4})$. This class is only well-defined up to the image of the Bockstein $H^3(B\,\mathrm{O}(A,q), \mathbf{k}^\times) \to H^4(B\,\mathrm{O}(A,q), \mu_{l^4})$, and we fix such a choice. In particular $\mathrm{3Pin}(A,q)$ and $\mathrm{3Spin}(A,q)$ depend on this choice.

Definition 4.48. Define the 3-group $\mathrm{3Pin}(A,q)$ as $\Omega\,\mathrm{hofib}(c)$.

We will henceforth write this class c as:

$$c(\mathrm{3Pin}) \in H^4(B\,\mathrm{O}(A,q), \mu_{l^4}) \ . \tag{4.49}$$

By definition, $c(\mathrm{3Pin})$ classifies $B\,\mathrm{3Pin}$ over $B\,\mathrm{O}(A,q)$. We have a diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & B^3\mu_{l^4} & \longrightarrow & B\,\mathrm{3Pin}(A,q) & \longrightarrow & B\,\mathrm{O}(A,q) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & B^3\mathbf{k}^\times & \longrightarrow & B\,\mathrm{3Lip}(A,q) & \longrightarrow & B\,\mathrm{O}(A,q) & \longrightarrow & 1 \end{array} \tag{4.50}$$

where the rows are exact.

Recall $\mathrm{SO}(A,q)$ from (4.16). The class $c(\mathrm{3Pin})$ in (4.49) pulls back to a class on $\mathrm{SO}(A,q)$, which we suggestively write as

$$c(\mathrm{3Spin}) \in H^4(B\,\mathrm{SO}(A,q), \mu_{l^4}) \ . \tag{4.51}$$

Definition 4.52. Define the 3-group $\mathrm{3Spin}(A,q)$ as $\Omega\,\mathrm{hofib}(c(\mathrm{3Spin}))$.

The following follows from the definition of 3Pin, 3Spin, and the universal property of the homotopy fiber.

Proposition 4.53. *The* 3*-group* $\mathrm{3Spin}(A,q)$ *is the pullback:*

$$\begin{array}{ccc} \mathrm{3Spin}(A,q) & \dashrightarrow & \mathrm{3Pin}(A,q) \\ \downarrow & & \downarrow \\ \mathrm{SO}(A,q) & \longrightarrow & \mathrm{O}(A,q) \end{array}$$

Remark 4.54. Note that if $O_4(A,q)$ vanishes, then Definitions 4.46, 4.48, and 4.52 are trivial (i.e. split) extensions of $\mathrm{O}(A,q)$ (resp. $\mathrm{SO}(A,q)$). E.g. if A is a vector space over $\mathbb{F}_p$ and q is the split quadratic form as discussed in Section 4.4.6.

4.6. **Interlude: spinors.** We collect some well-known facts about the Clifford algebra and spinors. We refer the reader to [Del99] for a more in-depth reference.

Let (V,q) be nondegenerate quadratic vector space over a field $\mathbf{k}$ with $\mathrm{char}\,\mathbf{k} \neq 2$. The *Clifford algebra* is following quotient of the tensor algebra:

$$\mathrm{Cliff}(V,q) \coloneqq T(V) / (v \otimes v = q(v)) \ .$$

One can calculate the anti-commutators in $\mathrm{Cliff}(V,q)$ to be:

$$uv + vu = b_q(u,v),$$

where b_q is the symmetric bihomomorphism corresponding to q (recall $\mathrm{char}\,k \neq 2$). Note that $\mathrm{Cliff}(V,q)$ is $\mathbb{Z}/2$-graded, with the image of V being odd. Write $p(a)$ for the parity of a homogeneous element. The tensor algebra has an anti-automorphism sending

$$v_1 \wedge \cdots \wedge v_N \mapsto v_N \wedge \cdots \wedge v_1$$

in degree N. The ideal we quotient out by to define Cliff is preserved by this anti-automorphism, so we obtain an antiautomorphism of Cliff. Write $a \mapsto a^T$ for this antiautomorphism.

Define the group Γ to be the subgroup of $\mathrm{Cliff}(V,q)^\times$ consisting of homogeneous elements which normalize the copy of V inside of $\mathrm{Cliff}(V,q)$. This is sometimes called the *Clifford group* or the *Lipschitz group*. Let $g \in \Gamma$ act on V by sending:

$$v \mapsto (-1)^{p(g)} gvg^{-1} \ .$$

This defines a map $\Gamma \to \mathrm{O}(V,q)$. Since $\mathrm{char}\, k \neq 2$, $\mathrm{O}(V,q)$ is generated by reflections (by the Cartan–Dieudonné theorem) so this map is onto. The kernel is given by scalars, so we have a short exact sequence

$$1 \to \mathbf{k}^\times \to \Gamma \to \mathrm{O}(V,q) \to 1 \ .$$

We cut the coefficients down to $\{\pm 1\} \hookrightarrow \mathbf{k}^\times$ as follows. The spinor norm is the map

$$\begin{aligned} \Gamma &\xrightarrow{N} \mathbf{k}^\times \\ g &\longmapsto gg^T \end{aligned}$$

Define the group:

$$\mathrm{Pin}(V,q) \coloneqq \ker(N) \ .$$

There is a unique map $N_O \colon \mathrm{O}(V) \to \mathbf{k}^\times / (\mathbf{k}^\times)^2$, which is also sometimes called the spinor norm, such that

$$\begin{array}{ccc} \Gamma & \longrightarrow & \mathrm{O}(V) \\ \downarrow{\scriptstyle N} & & \downarrow{\scriptstyle N_O} \\ \mathbf{k}^\times & \longrightarrow & \mathbf{k}^\times / (\mathbf{k}^\times)^2 \end{array}$$

This fits into the following diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \{\pm 1\} & \longrightarrow & \mathrm{Pin}(V,q) & \longrightarrow & \ker(N_O) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathbf{k}^\times & \longrightarrow & \Gamma & \longrightarrow & \mathrm{O}(V) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow{\scriptstyle N} & & \downarrow{\scriptstyle N_O} & & \\ 1 & \longrightarrow & (\mathbf{k}^\times)^2 & \longrightarrow & \mathbf{k}^\times & \longrightarrow & \mathbf{k}^\times / (\mathbf{k}^\times)^2 & \longrightarrow & 1 \end{array} \tag{4.55}$$

where the rows are short exact sequences. The spinor norm N_O is the obstruction to $\mathrm{Pin}(V)$ being a double cover of $\mathrm{O}(V)$: $\mathrm{Pin}(V)$ is automatically a double cover of $\ker(N_O)$, but if N_O is trivial then $\ker(N_O) = \mathrm{O}(V)$.

Example 4.56. If $\mathbf{k}$ is algebraically closed then $\mathbf{k}^\times / (\mathbf{k}^\times)^2$ is trivial, so the spinor norm is necessarily trivial. If $\mathbf{k} = \mathbb{R}$, then the spinor norm vanishes if q is positive definite. Therefore $\mathrm{Pin}(V)$ is a double cover of $\mathrm{O}(V)$ in these cases.

Define Spin to be the restriction of this extension to $\mathrm{SO} \subset \mathrm{O}(V)$. I.e. the following pullback.

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \{\pm 1\} & \longrightarrow & \mathrm{Pin}(V) & \longrightarrow & \mathrm{O}(V) & \longrightarrow & 1 \\ & & \| & & \uparrow & & \uparrow & & \\ 1 & \longrightarrow & \{\pm 1\} & \longrightarrow & \mathrm{Spin}(V) & \longrightarrow & \mathrm{SO}(V) & \longrightarrow & 1 \end{array}$$

For ease of exposition, assume $\dim V = 2n$ and $\mathbf{k} = \mathbb{C}$ (or $\mathbf{k} = \mathbb{R}$ with q of signature (n,n)) In this case the algebra $\mathrm{Cliff}(V,q)$ is the matrix algebra $M_{2^n}(\mathbf{k})$, and therefore has a unique simple module S up to isomorphism. Note that the automorphism group of the module S is $\mathbf{k}^\times$. Automorphisms of $\mathrm{Cliff}(V)$ only lift to automorphisms of the module S up to isomorphism, i.e. up to scalars. In particular this makes S a projective $\mathrm{O}(V,q)$-representation.

There exists an identification $V \simeq L \oplus L^*$, and a model for the simple module S is given by the exterior algebra $\wedge^\bullet L^*$. The algebra $\mathrm{Cliff}(V)$ can be identified with $\mathrm{End}(\wedge^\bullet L^*)$ by sending $(\ell, 0) \in V$ to $\ell \wedge (-)$, and $(0, \varphi) \in V$ to the interior product ι_φ.

4.7. **A detailed analogy with** Spin**.** See Section 4.6 where we collect some basic facts and notation involving the well-known spin-representation. In Table 1, we present an analogy between various objects in the theory of the spin-representation, and various objects introduced throughout Section 4.

The Clifford algebra associated to a quadratic vector space (V, q) is analogous to the braided fusion category associated to a metric group (A, q). One should think that this is a double-categorification: $\mathbb{E}_1$-algebras are being replaced with $\mathbb{E}_2$-categories. The Clifford algebra can be viewed as an associative deformation of the exterior algebra: the anticommutator in the Clifford algebra $\{x, y\} = q(x, y)$ becomes the usual multiplication on the exterior algebra (up to sign) for trivial q. Similarly, we can think that the braided fusion category $\mathcal{A}$ is a braided deformation of the symmetric fusion category $\mathbf{Vect}[A]$ with convolution: The braiding

$$\beta_q \colon \mathbf{k}_a * \mathbf{k}_b \xrightarrow{b(a,b)\,\mathrm{id}} \mathbf{k}_b * \mathbf{k}_a$$

becomes the symmetric braiding for trivial q.

The group $\mathrm{O}(V, q)$ plays the same role as $\mathrm{O}(A, q)$, and they both contain their respective special orthogonal subgroups as the kernel of the determinant. The ordinary group of scalars $\mathbf{k}^\times$ is replaced by the 3-group $B^2\mathbf{k}^\times$. Again we have gone up two categorical levels from an ordinary group to a 3-group. The group of automorphisms of $\mathrm{Cliff}(V)$ contains the affine orthogonal group $V \rtimes \mathrm{O}(V, q)$. As long as A is of odd order, the entire 2-group $\mathrm{Aut}_{\mathbf{EqBr}}(\mathcal{A})$ is a semidirect product of BA and $\mathrm{O}(A, q)$. This is one instance where things become easier upon categorification. See Remark 4.58.

The Clifford/Lipschitz group $\Gamma = \mathrm{Lip}(V, q)$ is a central extension of $\mathrm{O}(V, q)$ by $\mathbf{k}^\times$, just as $3\mathrm{Lip}(A, q)$ is a central extension of $\mathrm{O}(A, q)$ by $B^2\mathbf{k}^\times$ (Remark 4.41). The cohomology class $O_4(A, q)$ is of order dividing l^4 (where $l = |L| = \sqrt{|A|}$), whereas Γ is induced by a double cover. I.e. $B^2\mu_{l^4}$ plays the role of μ_2. The 3-group 3Pin (resp. 3Spin) is the analogue of Pin (resp. Spin). This is a difference between the two sides of the analogy: 3Pin and 3Spin are not higher double covers (i.e. extensions by $B^2\mu_2$) even though Pin and Spin are double covers.

Remark 4.57. Another aspect of the spin group is that (in signature (m, n) with either m or n being ≤ 1) it is the universal cover of SO. However, outside of these cases the Spin groups have fundamental group $\mathbb{Z}/2$. In particular, the split case $\mathrm{Spin}(n, n)$ is not simply-connected, and this is the case which is most directly analogous to the 3-dimensional setting: $q = \mathrm{ev}$ (for A a vector space over $\mathbb{F}_p$) has signature (n, n).

The module $S = \wedge^\bullet L^*$ is analogous to the fusion category $\mathcal{C} = \mathbf{Vect}[L^*]$ with convolution. This is again a double-categorification: We have replaced $\mathbb{E}_0$-algebras with $\mathbb{E}_1$-categories. Just as S is a module over $\mathrm{Cliff}(V)$, $\mathcal{C}$ is a module over $\mathcal{A}$, since $\mathcal{A} \cong \mathcal{Z}(\mathcal{C})$.

The 3-group $\mathrm{Pic}(\mathcal{A})$ is identified with invertible $\mathcal{C}$-bimodules,

$$\mathrm{Pic}(\mathcal{A}) \simeq \mathrm{Aut}_{\mathbf{Fus}}(\mathcal{C}) \ ,$$

in an analogous way in which the Clifford algebra itself is identified with the endomorphisms of spinors. Therefore Γ (resp. Spin) acts on S just as 3Lip (resp. 3Spin) automatically acts on $\mathcal{C}$.

Remark 4.58. This suggests that the analogue of $\mathrm{Pic}(\mathcal{A})$ should be the invertible part of the Clifford algebra. This is not entirely the case: an important fact about $\mathrm{Pic}(\mathcal{A})$ is that it is an extension of $\mathrm{Aut}_{\mathbf{EqBr}}(\mathcal{A})$ by $B^2\mathbf{k}^\times$. There is a more analogous object in the setting of metaplectic quantization, rather than spin quantization. There is a well-known analogy between these two settings: instead of starting with a quadratic vector space, we start with a symplectic vector space. The Clifford

algebra is replaced by the Weyl algebra, and the analogue of the Spin double-cover of SO is the double cover Mp of Sp. See [Del99] for a more in-depth explanation of the analogy.

The analogue of $\mathrm{Aut}_{\mathbf{EqBr}}(\mathcal{A})$ in the metaplectic setting is what is sometimes called the *affine symplectic group*:

$$A\,\mathrm{Sp}(V,\omega) = V \rtimes \mathrm{Sp}(V,\omega) \ .$$

Then the *extended symplectic group* $E\,\mathrm{Sp}$ is an extension of $A\,\mathrm{Sp}$ by $\mathrm{U}(1)$, and in fact

$$E\,\mathrm{Sp}(V,\omega) = \mathrm{Heis} \rtimes \mathrm{Mp}(V,\omega) \ ,$$

where Heis is the Heisenberg Lie group associated to (V,ω). I.e. we have a diagram:

$$\begin{array}{ccccc}
\mathrm{U}(1) & \longrightarrow & \mathrm{Heis} & \longrightarrow & V \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{U}(1) & \longrightarrow & E\,\mathrm{Sp}(V,\omega) & \longrightarrow & A\,\mathrm{Sp}(V,\omega) \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{Z}/2 & \longrightarrow & \mathrm{Mp} & \longrightarrow & \mathrm{Sp}
\end{array}$$

with exact rows and columns. This is analogous to the diagram:

$$\begin{array}{ccccc}
B^3\mathbf{k}^\times & \longrightarrow & B^2\mathcal{A}^\times & \longrightarrow & B^2 A \\
\downarrow & & \downarrow & & \downarrow \\
B^3\mathbf{k}^\times & \longrightarrow & B\,\mathrm{Pic}(\mathcal{A}) & \longrightarrow & B\,\mathrm{Aut}_{\mathbf{EqBr}}(\mathcal{A}) \\
\downarrow & & \downarrow & & \downarrow \\
B^3\mu_{l^4} & \longrightarrow & B\,3\mathrm{Pin} & \longrightarrow & B\,\mathrm{O}(A,q)
\end{array}$$

There is an analogue of the Heisenberg Lie algebra in the Spin setting: It is the super Lie algebra generated by V with super-bracket determined by the quadratic form. The analogue of Heis should be some super group integrating this super Lie algebra.

Note that, even though $V \rtimes \mathrm{Sp}(V)$ acts on the Weyl algebra, it is not necessarily the whole automorphism group. I.e. see [BKK05] where the question of identifying the full automorphism group is discussed.

5. Projective 3-dimensional TQFTs

5.1. **Dualizability and invertibility.** It was shown in [DSPS20] that the category of fusion categories **Fus** has duals.[11] Therefore the Cobordism Hypothesis tells us that, given a fusion category $\mathcal{C}$, there is a uniquely defined fully-extended topological field theory

$$\begin{array}{ccc}
\mathbf{Bord}_3^{\mathrm{fr}} & \xrightarrow{F} & \mathbf{Fus} \\
* & \longmapsto & \mathcal{C}
\end{array} \tag{5.1}$$

This is sometimes called the *Turaev-Viro theory* associated to a fusion category [TV92, FT21].

It was shown in [BJS21] that the category of braided fusion categories **BrFus** has duals. Therefore, given a braided fusion category $\mathcal{A}$, the Cobordism Hypothesis provides a fully-extended topological field theory $\mathbf{Bord}_4^{\mathrm{fr}} \to \mathbf{BrFus}$ sending the point to $\mathcal{A}$. This is a framed version of the *Crane-Yetter theory* associated to $\mathcal{A}$ [CY93, FT21].

[11] Technically our definition of **Fus** differs from the setting of finite semisimple abelian categories by an Ind-completion. However, as discussed in Remark 4.1, we can nonetheless apply the theorems of [DSPS20, ENO10] in our context.

Remark 5.2. For ease of discussion, let $\mathcal{A}$ be the braided fusion category attached to a metric group (A, q). Consider the space of invertible objects in $\mathcal{A}$, written $\mathcal{A}^\times$. The fact that $\mathcal{A}$ was braided means we are able to deloop twice, to obtain a 3-type $B^2\mathcal{A}^\times$. The homotopy groups are all trivial except $\pi_2 = A$ and $\pi_3 = \mathbf{k}^\times$. The only remaining information needed to specify this space is the k-invariant in $H^4\left(B^2A, \mathbf{k}^\times\right)$. It is a theorem of Eilenberg-Mac Lane [EL54, Theorem 26.1] that there is an isomorphism $H^4\left(B^2A, \mathbf{k}^\times\right) \cong \mathrm{Quad}\left(A, \mathbf{k}^\times\right)$ (see (4.18)); write τ_q for the class corresponding to q.

Assuming Hypothesis Q, we can construct the theory associated to the π-finite space B^2A twisted by a cocycle representing τ_q, which turns out to be precisely the framed fully-extended Crane-Yetter theory from above sending the point to $\mathcal{A}$. This is discussed in Example B.9 (iv).

5.1.1. *Invertibility.* Invertibility of a braided tensor category was shown in [BJSS21] to be equivalent to checking three conditions. When the braided tensor category is fusion, these conditions become equivalent so we only have to check one. One of the conditions, *non-degeneracy*, asks if the *Müger center*[12] is trivial.

If $\mathcal{C}$ is a fusion category such that $\mathcal{A} \cong \mathcal{Z}(\mathcal{C})$, then the Müger center is known to be trivial [Müg03, DGNO10], so the TQFT sending the point to $\mathcal{Z}(\mathcal{C})$ is in fact invertible.

5.2. **WRT theories.** Recall the description of the projective target category $\mathbb{P}\mathbf{Fus}$ from Section 2.3.3, and the corresponding notion of a projective TQFT from Definition 2.17. Let $\mathcal{A}$ be any nondegenerate braided fusion category. Note that the regular module $\mathcal{A}_\mathcal{A}$ defines a 1-morphism in $\mathbf{BrFus}$ from $\mathcal{A}$ to 1, and it is fully-dualizable as an object of the arrow category [Haï24]. Since $\mathcal{A}$ is fusion and nondegenerate, it is also an invertible object of $\mathbf{BrFus}$ [BJSS21]. In other words, the 1-morphism $\mathcal{A}_\mathcal{A} \colon \mathcal{A} \to 1$ is in the subcategory $\mathbb{P}\mathbf{Fus} \subset \mathbf{BrFus}^{\downarrow 1}$.

Summarizing this discussion, the cobordism hypothesis implies that $\mathcal{A}$ defines a fully-extended projective TQFT:

$$\mathbf{WRT}_\mathcal{A} \colon \mathbf{Bord}_3^{\mathrm{fr}} \to \mathbb{P}\mathbf{Fus}\ .$$

This is fully-extended anomalous Witten-Reshetikhin-Turaev (WRT)[13] theory, with a caveat: it is not oriented. This would be the data of a homotopy $\mathrm{SO}(3)$-fixed point structure on this object of $\mathbb{P}\mathbf{Fus}$.

The anomaly of $\mathbf{WRT}_\mathcal{A}$ is, by definition, the TQFT:

$$\mathbf{CY}_\mathcal{A} \colon \mathbf{Bord}_4^{\mathrm{fr}} \to \mathbf{BrFus}$$

which is classified by sending the point to $\mathcal{A}$. Recall this is a framed version of the *Crane-Yetter theory* associated to $\mathcal{A}$ [CY93, FT21].

As it turns out, $\mathbf{WRT}_\mathcal{A}$ has a canonical projective action of the 2-group of braided autoequivalences of $\mathcal{A}$.

Theorem 5.3. *Let $\mathcal{A}$ be a nondegenerate braided fusion category, and consider a monoidal functor $\Phi \colon G \to \mathrm{Aut}_{\mathbf{EqBr}}(\mathcal{A})$. There is an anomalous TQFT*

$$\mathbf{WRT}_\mathcal{A}^G \colon \mathbf{Bord}_3^{BG} \to \mathbb{P}\mathbf{Fus}$$

which agrees with $\mathbf{WRT}_\mathcal{A}$ upon restriction to trivial G-bundles:

$$\begin{array}{ccc} \mathbf{Bord}^{BG} & & \\ \uparrow & \searrow^{\mathbf{WRT}_\mathcal{A}^G} & \\ \mathbf{Bord}^{fr} & \xrightarrow{\mathbf{WRT}_\mathcal{A}} & \mathbb{P}\mathbf{Fus} \end{array}$$

[12] The Müger center of a braided tensor category consists of the objects which braid trivially with all other objects.

[13] Oftentimes WRT theory denotes the linear TQFT defined out of the extended bordism category as in Remark 3.10. As is explained there, these are equivalent.

Furthermore, the anomaly theory of $\mathbf{WRT}^G_{\mathcal{A}}$,

$$\mathbf{CY}^G_{\mathcal{A}} \colon \mathbf{Bord}_4^{BG} \to \mathbf{BrFus} \ ,$$

agrees with $\mathbf{CY}_{\mathcal{A}}$ *upon restriction to trivial* G*-bundles.*

Proof. Symmetric monoidal functors from $\mathbf{Bord}_3^{B\,\mathrm{Aut}_{\mathbf{EqBr}}(\mathcal{A})}$ to $\mathbb{P}\mathbf{Fus}$ which restrict to $\mathbf{WRT}_{\mathcal{A}}$ are classified by maps:

$$B\,\mathrm{Aut}_{\mathbf{EqBr}}\,(\mathcal{A}) \to B\,\mathrm{Aut}_{\mathbb{P}\mathbf{Fus}}\left(1 \xrightarrow{\mathcal{A}_{\mathcal{A}}} \mathcal{A}\right) \ .$$

There is a canonical functor:

$$\mathrm{Aut}_{\mathbf{EqBr}}\,(\mathcal{A}) \to \mathrm{Aut}_{\mathbf{BrFus}}\,(\mathcal{A})$$

given by sending a braided autoequivalence φ to the identity bimodule twisted by φ, written ${}_{\mathcal{A}}\mathcal{A}_{\varphi(\mathcal{A})}$. Precomposing with the given monoidal functor $\Phi \colon G \to \mathrm{Aut}_{\mathbf{EqBr}}\,(\mathcal{A})$ defines an action:

$$\overline{\Phi} \colon G \to \mathrm{Aut}_{\mathbf{BrFus}}\,(\mathcal{A}) \ .$$

Now notice that Φ and $\overline{\Phi}$ assemble together into a functor

$$G \to \mathrm{Aut}_{\mathbb{P}\mathbf{Fus}}\,(\mathcal{A}_{\mathcal{A}}) \tag{5.4}$$

which on objects sends $g \in G$ to the lax square

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\overline{\Phi}(g)} & \mathcal{A} \\ {\scriptstyle \mathcal{A}_{\mathcal{A}}}\downarrow & \swarrow_{\Phi(g)} & \downarrow{\scriptstyle \mathcal{A}_{\mathcal{A}}} \\ \mathbf{Vect} & = & \mathbf{Vect} \end{array}$$

Here the 2-morphism filling the square is the equivalence of $\mathcal{A}$-modules $\mathcal{A}_{\mathcal{A}} \simeq \mathcal{A}_{\mathcal{A}} \otimes_{\mathcal{A}} {}_{\mathcal{A}}\mathcal{A}_{\Phi(g)(\mathcal{A})}$ given by the braided autoequivalence $\Phi\,(g)$ itself. The monoidal structure of Φ provides the coherence data making this assignment compatible with composition in G. Note that this classifies a functor

$$\mathbf{WRT}^G_{\mathcal{A}} \colon \mathbf{Bord}_3^{BG} \to \mathbb{P}\mathbf{Fus} \ ,$$

and its restriction to trivial bundles sends the point to the regular module, and therefore agrees with $\mathbf{WRT}_{\mathcal{A}}$ by the cobordism hypothesis. The anomaly theory $\mathbf{CY}^G_{\mathcal{A}}$ is the composition of $\mathbf{WRT}^G_{\mathcal{A}}$ with the functor $\mathbb{P}\mathbf{Fus} \to \mathbf{BrFus}^{\times}$ from Definition 2.17, so its restriction to trivial G-bundles is the composition of this functor with $\mathbf{WRT}_{\mathcal{A}}$, i.e. $\mathbf{CY}_{\mathcal{A}}$. □

Referee 2 (#15) asked what it means to put $\Phi(g)$ on the diagonal. The diagonal was a 1-morphism arrow $\mathbf{Vect} \to \mathcal{A}$; it is now a 2-cell, and the preceding sentence says which one. Check the direction of this 2-cell against the lax convention of [JFS17] (top arrow ${}_{\mathcal{A}}\mathcal{A}_{\Phi(g)(\mathcal{A})}$, bottom arrow $\mathrm{id}_{\mathbf{Vect}}$); flip if the square was intended the other way round.

Remark 5.5. Since $\mathcal{A}$ is nondegenerate, it is invertible in the 4-category $\mathbf{BrFus}$, and therefore we have equivalences:

$$\mathrm{Aut}_{\mathbf{BrFus}}\,(\mathcal{A}) \simeq \mathrm{Aut}_{\mathbf{BrFus}}\,(1) \simeq \mathbf{Fus}^{\times} \simeq B^3\mathbf{k}^{\times} \ .$$

The delooping of this map pulls back along the monoidal functor $G \to \mathrm{Aut}_{\mathbf{EqBr}}\,(\mathcal{A})$ to a 4-cocycle on G, which classifies the anomaly theory $\mathbf{CY}^G_{\mathcal{A}}$ as a functor out of $\mathbf{Bord}_4^{BG}$.

Remark 5.6. Recall from Remark 3.10 that, given an anomalous TQFT, there is an extended bordism category over which the anomaly can always be trivialized. As is explained in [Wal06, Fre12, BDSPV15, Haï25], if you lift from the oriented 4-dimensional bordism category to the bordism category of oriented 4-manifolds equipped with a trivialization of p_1, then the Crane-Yetter theory can be trivialized. Another option is equipping the 4-manifolds with a signature structure. Choices of the latter structure correspond to square roots of the central charge, and the choices of the former correspond to sixth roots, so these relate to one another by a factor of three [BDSPV15].

Remark 5.7. If the anomaly theory happens to be trivializable as a functor out of $\mathbf{Bord}^{BG}$, then the trivializations $1 \xrightarrow{\sim} \mathbf{CY}^G$ will form a torsor over invertible theories of one dimension lower. We will save a more detailed discussion of trivializations for the more restricted examples of 3-dimensional theories in the coming sections.

5.3. **The anomaly and the center.** Consider the following theory ζ.

$$\begin{array}{ccc} \mathbf{Bord}_4^{\mathrm{fr}} & \xrightarrow{\ \zeta\ } & \mathbf{BrFus} \\ * & \longmapsto & \mathcal{Z}(\mathcal{C}) \end{array}$$

Note that $\mathcal{C}$ always defines a module ${}_{\mathcal{Z}}\mathcal{C}$ over its own center $\mathcal{Z}(\mathcal{C})$. By [JFS17, Theorem 7.15], this morphism ${}_{\mathcal{Z}}\mathcal{C}\colon 1 \to \mathcal{Z}(\mathcal{C})$ in **BrFus** classifies a relative theory, i.e. a lax natural transformation (Remark A.11):

$$F_\zeta\colon 1 \to \zeta\ .$$

By Section 5.1.1, ζ is invertible, so we can think of F_ζ as an anomalous theory.

In fact, the Drinfeld center of a fusion category $\mathcal{C}$ is trivial in **BrFus**: $\mathcal{C}$ defines an equivalence between 1 and $\mathcal{Z}(\mathcal{C})$. See the proof of [BJSS21, Theorem 4.2]. For a general braided fusion category $\mathcal{A}$, an $\mathcal{A}$-central fusion category $\mathcal{C}$ was shown to give an equivalence between $\mathcal{A}$ and **Vect** if and only if the natural map $\mathcal{A} \to \mathcal{Z}(\mathcal{C})$ is an equivalence in [JMPP22, Theorem 2.23].

Remark 5.8. Recall $\mathrm{Pic}(\mathcal{Z}(\mathcal{C}))$ from Section 4.2. Note that these modules are not 1-morphisms in **BrFus** (they are only modules, not central modules). They are however 2-morphisms in **BrFus**. In the language of defects, this is saying that Pic does not consist of domain walls, but rather certain codimension-2 defects. In particular, consider a domain wall between the trivial theory and ζ labelled by $\mathcal{Z}(\mathcal{C})$ as a module over itself. Now the objects of Pic label self-interfaces between this domain wall.

5.4. **Projective action of the braided automorphisms of the center.** Fix a TQFT F associated to $\mathcal{C} \in \mathbf{Fus}$ as in (5.1). We will apply Corollary 3.17 to this theory.

Recall the notation from Section 4.4.4. In particular, recall the class from (4.39):

$$k_3\colon B\,\mathrm{Aut}_{\mathbf{EqBr}}(\mathcal{Z}(\mathcal{C})) \to B^4\mathbf{k}^\times\ .$$

In the notation of Section 3.2, we will let $X = B\,\mathrm{Aut}_{\mathbf{EqBr}}(\mathcal{Z}(\mathcal{C}))$ and $c = k_3$, so

$$\widetilde{X} = B\,\mathrm{Aut}_{\mathbf{Fus}}(\mathcal{C})\ .$$

Since $\pi_3 B\,\mathrm{Aut}_{\mathbf{Fus}}(\mathcal{C}) \cong \mathbf{k}^\times$ (see Section 4.4.3), $\mathcal{C}$ is a Schur object (Definition 3.14), and therefore Corollary 3.17 applies.

By the general discussion of Section 3.2, there is an anomaly theory, i.e. a once categorified 3-dimensional TQFT

$$\alpha_{k_3}\colon \mathbf{Bord}_3^{B\,\mathrm{Aut}_{\mathbf{EqBr}}(\mathcal{Z}(\mathcal{C}))} \to \mathbf{BrFus}\ ,$$

classified by k_3.

Remark 5.9. This is a situation where the anomaly theory α_{k_3} actually extends to a $(d+1) = 4$-dimensional theory, i.e. a functor out of $\mathbf{Bord}_4^{B\,\mathrm{Aut}_{\mathbf{EqBr}}(\mathcal{Z}(\mathcal{C}))}$, by construction (Section 3.2).

Theorem 5.10. *The framed TQFT F has an $X = B\,\mathrm{Aut}_{\mathbf{EqBr}}(\mathcal{Z}(\mathcal{C}))$-anomaly as in Definition 3.4. In particular, the fusion category $\mathcal{C}$ itself defines an anomalous theory:*

$$F_{k_3}\colon \alpha_{k_3} \to 1_{B\,\mathrm{Aut}_{\mathbf{EqBr}}(\mathcal{Z}(\mathcal{C}))}\ . \tag{5.11}$$

Furthermore, if $\mathcal{Z}(\mathcal{C})$ is pointed, and the cohomology class classifying the braiding of $\mathcal{Z}(\mathcal{C})$ (as in Remark 5.2) is nontrivial (e.g. whenever $\mathcal{C} \not\simeq \mathbf{Vect}$), then the class k_3 classifying the anomaly α_{k_3} is nontrivial.

Proof. As remarked above, $\mathcal{C}$ is a Schur object (Definition 3.14) of **Fus**, so Corollary 3.17 applies. The fact that $F\colon * \mapsto \mathcal{C}$ has an X-anomaly is then Corollary 3.17(ii).

Recall from Remark 5.2 that the class classifying the braiding is the same as the k-invariant of $B^2\mathcal{Z}^\times$. Also recall the Postnikov and Whitehead towers of $B\,\mathrm{Aut}_{\mathbf{Fus}}(\mathcal{C})$ from (4.31). The

composition of k_3 with the map from the universal cover agrees with the k-invariant of $B^2\mathcal{Z}^\times$ over $B^2\pi_2$:

$$\begin{array}{ccc} B^2\mathcal{Z}(\mathcal{C})^\times & \longrightarrow & B\operatorname{Aut}_{\mathbf{Fus}}(\mathcal{C}) \\ \downarrow & & \downarrow \\ B^2\pi_2 & \longrightarrow & B\operatorname{Aut}_{\mathbf{EqBr}}(\mathcal{Z}(\mathcal{C})) \\ & \searrow^{\tau_q} & \searrow^{k_3} \\ & & B^4\mathbf{k}^\times \end{array}$$

where we write τ_q for the class corresponding to q as in Remark 5.2. From this diagram we see that trivial k_3 implies trivial τ_q. Therefore if τ_q is nontrivial, then k_3 is nontrivial. □

5.4.1. *Trivializing the anomaly.* Recall the setting of Section 4.4.4. If we can trivialize k_3 over

$$f_2 \colon G_2 \to B\operatorname{Aut}_{\mathbf{EqBr}}(\mathcal{Z}(\mathcal{C}))\ ,$$

then the trivializations form a torsor over $H^3(BG_2, \mathbf{k}^\times)$, and a single trivialization determines

$$1_{BG_2} \xrightarrow{\sim} \alpha_{f_2 \circ c} \to 1_{BG_2}\ ,$$

which is a (non-anomalous) G_2-theory by Proposition A.9. This is Corollary 3.17(b). The equivalence with Corollary 3.17(c) and (d) in this example will be discussed in Sections 5.6 and 5.8.

5.5. **Projective action of the orthogonal group.** Still fix $F \colon * \mapsto \mathcal{C}$ (5.1), as in Sections 5.1, 5.3, and 5.4. Although now we will assume that we are in the setting of Section 4.4.3. I.e. we are assuming that the obstruction $O_3(f)$ from (4.32), associated to a map

$$f \colon G \to \pi_1 B\operatorname{Aut}_{\mathbf{Fus}}(\mathcal{C})\ ,$$

is trivializable, and that we have chosen a trivialization s. Recall the class

$$O_4(f,s) \in H^4\left(BG, \mathbf{k}^\times\right)$$

from (4.37).

We will apply Theorem 3.12 to this setting. I.e. let $\mathcal{T} = \mathbf{BrFus}$, as in Section 5.4. Set $X = BG$ and $c = O_4(f,s)$. By the general discussion of Section 3.2, there is an anomaly theory

$$\alpha_{O_4(f,s)} \colon \mathbf{Bord}_3^{BG} \to \mathbf{BrFus} \tag{5.12}$$

classified by $O_4(f,s)$.

By definition, the space classified by $O_4(f,s)$ is the pullback:

$$\begin{array}{ccc} \operatorname{hofib}(O_4(f,s)) & \longrightarrow & B\operatorname{Aut}_{\mathbf{Fus}}(\mathcal{C}) \\ \downarrow & & \downarrow \\ BG & \xrightarrow{s} & \pi_{\leq 2} B\operatorname{Aut}_{\mathbf{Fus}}(\mathcal{C}) \end{array}$$

The inclusion of the connected component corresponding to $\mathcal{C}$ defines a map:

$$B\operatorname{Aut}_{\mathbf{Fus}}(\mathcal{C}) \to \mathbf{Fus}\ ,$$

so composition with the map defined by the pullback gives us:

$$\operatorname{hofib}(O_4(f,s)) \to B\operatorname{Aut}_{\mathbf{Fus}}(\mathcal{C}) \to \mathbf{Fus}\ ,$$

which classifies a TQFT (by the Cobordism Hypothesis (A.6)):

$$\mathbf{Bord}_3^{\operatorname{hofib}(O_4(f,s))} \to \mathbf{Fus}\ .$$

This theory is the extended TQFT associated to the anomalous theory

$$F_{O_4(f,s)} \colon \alpha_{O_4(f,s)} \to 1_{BG}$$

by Proposition 3.8.

5.5.1. *Pointed center.* Continuing the investigation of TQFTs of the form (5.1), we now assume that $\mathcal{Z}(\mathcal{C})$ is pointed. As in Proposition 4.29, this is classified by its group A of simple objects and a quadratic form $q\colon A \to \mathbf{k}^\times$ which describes the braiding.

We assume $|A|$ is odd, so that by Section 4.4.5 there is a canonical splitting s of $B\operatorname{Aut}_{\mathbf{EqBr}}(\mathcal{Z}(\mathcal{C}))$. Therefore there is a universal class defined in (4.40):

$$O_4 = O_4(A,q) \in H^4\left(B\,\mathrm{O}(A,q), \mathbf{k}^\times\right) \ .$$

Write $\mathrm{O}(A) = \mathrm{O}(A,q)$ for brevity.

By the general discussion in Section 3.2, there is an anomaly classified by O_4:

$$\alpha_{O_4}\colon \mathbf{Bord}^{B\,\mathrm{O}(A)} \to \mathbf{BrFus} \ . \tag{5.13}$$

Recall the 3-group $3\mathrm{Lip}(A)$ defined in Section 4.5. By definition, the space classified by O_4 is the pullback:

$$\begin{array}{ccc} B\,3\mathrm{Lip}(A) = \mathrm{hofib}(O_4) & \longrightarrow & B\operatorname{Aut}_{\mathbf{Fus}}(\mathcal{C}) \\ \downarrow & & \downarrow \\ B\,\mathrm{O}(A) & \xrightarrow{\ s\ } & \pi_{\leq 2} B\operatorname{Aut}_{\mathbf{Fus}}(\mathcal{C}) \end{array}$$

where s is the canonical splitting

$$s\colon B\,\mathrm{O}(A) \to B\operatorname{Aut}_{\mathbf{EqBr}}(\mathcal{Z}(\mathcal{C})) \simeq \pi_{\leq 2} B\operatorname{Aut}_{\mathbf{Fus}}(\mathcal{C}) \ .$$

The inclusion of the connected component corresponding to $\mathcal{C}$ defines a map:

$$B\operatorname{Aut}_{\mathbf{Fus}}(\mathcal{C}) \to \mathbf{Fus} \ ,$$

so composition gives us a map:

$$B\,3\mathrm{Lip}(A) \to B\operatorname{Aut}_{\mathbf{Fus}}(\mathcal{C}) \to \mathbf{Fus} \ , \tag{5.14}$$

which classifies a TQFT:

$$F_{3\mathrm{Lip}}\colon \mathbf{Bord}_3^{B\,3\mathrm{Lip}(A)} \to \mathbf{Fus}$$

by the Cobordism Hypothesis (A.6). This is the extended theory corresponding to the anomalous theory

$$F_{O_4}\colon \alpha_{O_4} \to 1_{BG}$$

by Proposition 3.8.

The same holds when we restrict to the 3-groups $3\mathrm{Pin}(A)$ and $3\mathrm{Spin}(A)$ defined in Section 4.5 as follows. In particular, recall the cohomology classes (4.49) and (4.51):

$$c(3\mathrm{Pin}) \in H^4\left(B\,\mathrm{O}(A,q), \mu_{l^4}\right) \qquad c(3\mathrm{Spin}) \in H^4\left(B\,\mathrm{SO}(A,q), \mu_{l^4}\right) \ .$$

These classify anomaly theories, by the general discussion in Section 3.2:

$$\alpha_{c(3\mathrm{Pin})}\colon \mathbf{Bord}_3^{B\,\mathrm{O}(A)} \to \mathbf{BrFus} \tag{5.15}$$

$$\alpha_{c(3\mathrm{Spin})}\colon \mathbf{Bord}_3^{B\,\mathrm{SO}(A)} \to \mathbf{BrFus} \ . \tag{5.16}$$

We have a map, e.g. from (4.50), $3\mathrm{Pin}(A) \to 3\mathrm{Lip}(A)$. We also have a map $3\mathrm{Spin}(A) \to 3\mathrm{Pin}(A)$ from Proposition 4.53, which we can compose with the map $3\mathrm{Pin}(A) \to 3\mathrm{Lip}(A)$ to obtain a map $3\mathrm{Spin}(A) \to 3\mathrm{Lip}(A)$. The upshot of this, is that we can compose these maps with (5.14) to obtain functors:

$$3\mathrm{Pin}(A) \to \mathbf{Fus} \qquad\qquad 3\mathrm{Spin}(A) \to \mathbf{Fus}$$

which classify a 3Pin(A)-TQFT and a 3Spin(A)-TQFT:

$$F_{3\mathrm{Pin}}\colon \mathbf{Bord}^{B\,3\mathrm{Pin}(A)} \to \mathbf{Fus}$$
$$F_{3\mathrm{Spin}}\colon \mathbf{Bord}^{B\,3\mathrm{Spin}(A)} \to \mathbf{Fus}$$

by the Cobordism Hypothesis (A.6). These are the extended theories associated to the anomalous theories:

$$F_{c(3\mathrm{Pin})}\colon \alpha_{c(3\mathrm{Pin})} \to 1_{B\,\mathrm{O}(A)}$$
$$F_{c(3\mathrm{Spin})}\colon \alpha_{c(3\mathrm{Spin})} \to 1_{B\,\mathrm{SO}(A)}$$

by Proposition 3.8, where $1_{B\,\mathrm{O}(A)}$ (resp. $1_{B\,\mathrm{SO}(A)}$) denotes the trivial **BrFus**-valued once-categorified theory defined on $\mathbf{Bord}_3^{B\,\mathrm{O}(A)}$ (resp. $\mathbf{Bord}_3^{B\,\mathrm{SO}(A)}$).

Summarizing this discussion, we have the following.

Theorem 5.17. *Let $\mathcal{C}$ be a fusion category with pointed Drinfeld center. Recall this is classified by a polarized metric group (A, q, L) as in Proposition 4.29, and assume $|A|$ is odd. The nonanomalous framed theory*

$$F\colon \mathbf{Bord}_3^{fr} \to \mathbf{Fus}$$

sending the point to $\mathcal{C}$ has an $\mathrm{O}(A, q)$-anomaly in the sense of Definition 3.4. I.e. there is an anomaly theory

$$\alpha_{c(3\mathrm{Pin})}\colon \mathbf{Bord}_3^{B\,\mathrm{O}(A,q)} \to \mathbf{BrFus}\ ,$$

and $\mathcal{C}$ canonically defines an anomalous $\mathrm{O}(A, q)$-TQFT:

$$F_{c(3\mathrm{Pin})}\colon \alpha_{c(3\mathrm{Pin})} \to 1_{B\,\mathrm{O}(A,q)}\ . \tag{5.18}$$

Remark 5.19. As in Section 5.4, the anomalies $\alpha_{O_4(f,s)}$, and therefore α_{O_4}, $\alpha_{c(3\mathrm{Pin}(A))}$, and $\alpha_{c(3\mathrm{Spin})}$, defined in (5.12), (5.13), (5.15), and (5.16) extend to $(d+1) = 4$-dimensional theories, i.e. functors out of $\mathbf{Bord}_4^{BG}$ (for $G = \mathrm{O}(A)$ or $\mathrm{SO}(A)$) by construction (Section 3.2).

Corollary 5.20. *Restricting the $\mathrm{O}(A, q)$-anomaly of Theorem 5.17 to $\mathrm{SO}(A, q)$ we obtain an $\mathrm{SO}(A, q)$-anomaly of F*

$$\alpha_{c(3\mathrm{Spin})}\colon \mathbf{Bord}_3^{B\,\mathrm{SO}(A,q)} \to \mathbf{BrFus}\ ,$$

and an anomalous $\mathrm{SO}(A, q)$-equivariant theory

$$F_{c(3\mathrm{Spin})}\colon \alpha_{c(3\mathrm{Spin})} \to 1_{B\,\mathrm{SO}(A,q)}\ . \tag{5.21}$$

5.5.2. *Trivializing the anomaly.* As in Section 3.2.1, we ask if we can trivialize the anomaly from Theorem 5.17 (resp. Corollary 5.20) after pulling back along a map $f\colon G \to \mathrm{O}(A)$ (resp. $f\colon G \to \mathrm{SO}(A)$) for G a finite group.

The map f defines a new anomaly theory $\alpha_{f^*c(3\mathrm{Pin})}$:

$$\begin{array}{ccc} \mathbf{Bord}_3^{BG} & & \\ \downarrow{\scriptstyle f} & \searrow^{\alpha_{f^*c(3\mathrm{Pin})}} & \\ \mathbf{Bord}_3^{B\,3\mathrm{Pin}(A)} & \xrightarrow[\alpha_{c(3\mathrm{Pin})}]{} & \mathbf{BrFus}\ . \end{array}$$

Trivializing the class $f^*c(3\mathrm{Pin})$ is equivalent to trivializing the anomaly $\alpha_{f^*c(3\mathrm{Pin})}$, which yields a (non-anomalous) G-theory by Proposition A.9

$$1_{BG} \to \alpha_{f^*c(3\mathrm{Pin})} \to 1_{BG}\ .$$

In addition to a fusion category $\mathcal{C}$, the data defining a theory $\mathbf{Bord}_3^{BG} \to \mathbf{Fus}$ includes of a coherent action of G on $\mathcal{C}$. This means each element $g \in G$ is assigned a bimodule M_g, along with equivalences between the composition $\mathcal{M}_h \otimes \mathcal{M}_g$ and $\mathcal{M}_{gh}$ (giving rise to vanishing O_3 as in

Section 4.4.2) and associators giving rise to vanishing O_4. (See [ENO10, §7,8] for details.) This makes

$$\bigoplus_{g\in G} M_g \tag{5.22}$$

itself a fusion category, which is called a G-extension of $\mathcal{C}$.

Remark 5.23. [ENO10, Theorem 1.3] associates a G-extension of $\mathcal{C}$ to a trivialization of O_3 and O_4. If we use this trivialization to trivialize α, the G-extension of [ENO10] is precisely (5.22) from the G-theory $1_{BG} \to \alpha_{O_4} \to 1_{BG}$.

5.6. **Projective 3d TQFTs.** In order to describe the projectivity captured by the anomalies in Theorems 5.10 and 5.17 and Corollary 5.20 in terms of projective theories, as in Theorem 3.12 and Corollary 3.17, we consider the projectivization (Definition 2.3) of **Fus** discussed in Section 2.3.3.

Definition 5.24. Projective 3-dimensional TQFTs with tangential structure (X,ζ) are symmetric monoidal functors

$$\mathbf{Bord}_3^{(X,\zeta)} \to \mathbb{P}\mathbf{Tens}\ ,$$

where $\mathbb{P}\mathbf{Tens}$ denotes the projectivization of $\Omega\mathbf{BrTens} \simeq \mathbf{Tens}$. We will primarily restrict our attention to theories factoring through the subcategory $\mathbb{P}\mathbf{Fus}$. The *projectivity* of $\overline{F}$ is the theory α (as in Definition 2.17):

$$\mathbf{Bord}_3 \xrightarrow{\overline{F}} \mathbb{P}\mathbf{Fus} \xrightarrow{s} \mathbf{BrFus}^{\times}\ .$$
$$\mathbf{Bord}_3 \xrightarrow{\alpha} \mathbf{BrFus}^{\times}$$

Let $\mathcal{C}$ be an object of the Morita 3-category of fusion categories. As in (5.1), this classifies a framed theory $F\colon * \mapsto \mathcal{C}$.

Corollary 5.25. *As in Theorem 5.10, write $X = B\operatorname{Aut}_{\mathbf{EqBr}}(\mathcal{Z}(\mathcal{C}))$, and let k_3 be the cocycle in (4.35).*

The anomalous theory F_{k_3} in (5.11) is equivalent to some projective theory:

$$\overline{F}\colon \mathbf{Bord}_3^X \to \mathbb{P}\mathbf{Fus}$$

with underlying theory F, in the sense of Definition 2.19.

Corollary 5.25 follows from Corollary 3.17(iii) in the context of Theorem 5.10.

Let $c(\mathrm{3Pin})$ and $c(\mathrm{3Spin})$ be the cocycles defined in (4.49) and (4.51), which classify the anomaly theories $\alpha_{c(\mathrm{3Pin})}$ and $\alpha_{c(\mathrm{3Spin})}$ from (5.15) and (5.16).

Corollary 5.26. *Let $\mathcal{Z}(\mathcal{C})$ be pointed, with underlying pre-metric group (A,q). The anomalous theory $F_{c(\mathrm{3Pin})}$ from (5.18) is equivalent to some projective theory:*

$$\overline{F}\colon \mathbf{Bord}_3^{B\,\mathrm{O}(A,q)} \to \mathbb{P}\mathbf{Fus}$$

with underlying theory F, in the sense of Definition 2.19.

The same result holds when $F_{c(\mathrm{3Pin})}$ is replaced with $F_{c(\mathrm{3Spin})}$ from (5.21), and $\mathrm{O}(A)$ is replaced with $\mathrm{SO}(A)$.

Corollary 5.26 follows from Theorem 3.12(i) in the context of Theorem 5.17 and Corollary 5.20.

As in the general discussion in Section 2.4 (namely Proposition 2.18 and Definition 2.17) a trivialization of any of the anomaly theories α_{k_3}, $\alpha_{c(\mathrm{3Pin})}$, or $\alpha_{c(\mathrm{3Spin})}$ determines a linearization of $\overline{F}$, which tautologically agrees with the trivialized anomalous theory $1 \xrightarrow{\sim} \alpha \xrightarrow{F_c} 1$. This is Theorem 3.12(i) (or Corollary 3.17(c)) in this context.

On the other hand, nontrivial cocycles on finite groups *always* describe anomalies of some 3-dimensional TQFT by the following.

Corollary 5.27. *For any nontrivial* $\pi\colon BG \to B^4\mathbf{k}^\times$*, there exists a* π*-finite space* X *and nontrivial anomaly TQFT*

$$\alpha_\pi\colon \mathbf{Bord}_4^X \to \mathbf{BrFus}$$

such that there exists a map $f\colon BG \to X$ *such that the pullback of the cocycle classifying* α *along* f *agrees with the cohomology class of* π*.*

Proof. Proposition 4.45 provides us with a nondegenerate braided fusion category $\mathcal{B}$ and $\rho\colon G \to \mathrm{Aut}_{\mathbf{EqBr}}(\mathcal{B})$ such that $(B\rho)^* k_3 = [\pi]$. Setting $X = B\,\mathrm{Aut}_{\mathbf{EqBr}}(\mathcal{B})$ and $f = B\rho$, the anomaly theory α_{k_3} classified by k_3 (as in Section 3.2) has the desired property. □

5.7. **Functorial assignment of TQFTs to a vector space over a finite field.** Let $L \simeq \mathbb{F}_p^n$ be a vector space over a finite field $\mathbb{F}_p$ ($p \neq 2$). The fusion category $\mathcal{C} = (\mathbf{Vect}[L], *)$ defines a theory:

$$F\colon \mathbf{Bord}_3^{\mathrm{fr}} \to \mathbf{Fus}\ .$$

Note that

$$\mathcal{Z}(\mathcal{C}) \simeq \mathbf{Vect}\left[\mathbb{F}_p^{2n}\right]$$

with braiding induced by the quadratic form q_{split} on $V \simeq \mathbb{F}_p^{2n}$ of signature (n,n). Write $\mathrm{O}(n,n;\mathbb{F}_p)$ for the split orthogonal group over $\mathbb{F}_p$. Recall this agrees with $\mathrm{O}(V, q_{\mathrm{split}})$ (Observation 4.44).

Recall from Section 4.4.6 that the obstruction $O_4\left(\mathbb{F}_p^{2n}, q_{\mathrm{split}}\right)$ is shown to vanish in this case in [EG18, Theorem 6.1]. Therefore we have the following by Theorem 5.17.

Corollary 5.28. *There is a canonically defined* $H^3(B\,\mathrm{O}(n,n;\mathbb{F}_p), \mathbf{k}^\times)$*-torsor of* $\mathrm{O}(n,n;\mathbb{F}_p)$*-TQFTs*

$$F_{O_{n,n}}\colon \mathbf{Bord}_3^{B\,\mathrm{O}(n,n;\mathbb{F}_p)} \to \mathbf{Fus}\ .$$

Recall the analogy between this categorical representation of $\mathrm{O}(n,n;\mathbb{F}_p)$ and the Weil representation of the metaplectic group in Remark 4.58. Corollary 5.28 can be thought as an analogue of the fact that the Weil representation splits over a finite field [GH09, GH11].

5.8. **Module structures on 3d TQFTs.** To describe the projectivity captured by the anomalies in Theorems 5.10 and 5.17 and Corollary 5.20 in terms of twisted module structures as in Theorem 3.12 and Corollary 3.17, we need to assume Hypothesis Q for our 4-dimensional target $\mathcal{T}$.

There is latitude in the choice of target. One option is to use $\mathcal{T} = \mathbf{BrFus}$. This target is sufficient to support our anomaly theories α_{k_3}, $\alpha_{c(3\mathrm{Pin})}$, and $\alpha_{c(3\mathrm{Spin})}$ from Theorems 5.10 and 5.17 and Corollary 5.20.

Definition 5.29. We say a 4-dimensional target $\mathcal{T}$ is **BrFus**-*admissible* if:

Name is a placeholder.

(a) $\Omega\mathcal{T} = \mathrm{End}_{\mathcal{T}}(1) \simeq \mathbf{Fus}$, and
(b) there is a fully-faithful functor $\mathbf{BrFus} \to \mathcal{T}$,

In this case we can regard the anomaly theories α_{k_3}, $\alpha_{c(3\mathrm{Pin})}$, and $\alpha_{c(3\mathrm{Spin})}$ from Theorems 5.10 and 5.17 and Corollary 5.20 as valued in $\mathcal{T}$.

Example 5.30. As is remarked in [DR18], one can define an $(\infty,4)$-category of algebras in the 3-category of finite semisimple 2-categories (with bimodules, linear 2-functions, natural transformations, and modifications). It is conjectured in [DR18] that 2-fusion categories are the fully-dualizable objects of this $(\infty,4)$-category.

Finite nonabelian topological 4-dimensional G-gauge theory is simply classified by sending the point to the fusion 2-category $2\mathbf{Vect}[G]$ of G-graded 2-vector spaces [DR18, Construction 2.1.13].

Let $\mathcal{C}$ be an object of the Morita 3-category of fusion categories. As in (5.1), this classifies a framed theory $F\colon * \mapsto \mathcal{C}$.

Corollary 5.31. *Assuming Hypothesis Q for a* **BrFus***-admissible target $\mathcal{T}$ (Definition 5.29), the framed theory F has a canonical $\left(\sigma^4_{B\operatorname{Aut}_{\mathbf{EqBr}}(\mathcal{Z}(\mathcal{C})),k_3},\rho\right)$-module structure.*

Proof. On account of the anomalous theory (5.11), Corollary 3.17 (iv) implies the result. □

Let G_2 be a finite 2-group (i.e. BG_2 is a π-finite 2-type), let $f\colon BG_2 \to X$ be a map, and assume we have chosen a trivialization of f^*k_3. Assuming Hypothesis Q, by Corollary 3.17(d), we can reduce (in the sense of Definition B.24) the canonical $\left(\sigma^4_{B\operatorname{Aut}_{\mathbf{EqBr}}(\mathcal{Z}(\mathcal{C})),k_3},\rho\right)$-module structure on F in Corollary 5.31 to a $(\sigma_{BG_2},\rho_{BG_2})$-module structure. This in particular yields a G_2-theory by Proposition B.14, which agrees with the G_2-theory obtained by trivializing the anomaly.

Corollary 5.32. *Assuming Hypothesis Q for a* **BrFus***-admissible target $\mathcal{T}$ (Definition 5.29), there is a $\left(\sigma_{B\,\mathrm{O}(A),c(3\mathrm{Pin})},\rho_{B\,\mathrm{O}(A),c(3\mathrm{Pin})}\right)$-module structure on F. Similarly, there is a $\left(\sigma_{B\,\mathrm{SO}(A),c(3\mathrm{Spin})},\rho_{B\,\mathrm{SO}(A),c(3\mathrm{Spin})}\right)$-module structure on F.*

Proof. On account of the anomalous theory (5.18), Theorem 3.12(iii) implies the result. The $3\mathrm{Spin}\,(A)$ version follows from Theorem 3.12(iii), for the anomalous theory (5.21). □

Let $f\colon BG \to \mathrm{O}\,(A)$ for a finite group G. Given a trivialization of $f^*c\,(3\mathrm{Pin})$, the $\left(\sigma_{B\,\mathrm{O}(A),c(3\mathrm{Pin})},\rho\right)$-module structure on F can be reduced to a (σ_{BG},ρ_{BG})-module structure as in Proposition B.23. This in particular yields a theory on $\mathbf{Bord}_3^{BG}$ by Proposition B.14 which agrees with the theory obtained by trivializing the anomaly by Theorem 3.13(iii). The same is true when $3\mathrm{Pin}\,(A)$ is replaced by $3\mathrm{Spin}\,(A)$ and $f\colon G \to \mathrm{O}\,(A)$ is replaced by $f\colon G \to \mathrm{SO}\,(A)$.

Corollary 5.33. *Assuming Hypothesis Q for $\mathcal{T} = 2\mathbf{Fus}$, there is an $H^3\left(B\,\mathrm{O}\,(n,n;\mathbb{F}_p),\mathbf{k}^\times\right)$-torsor of $\left(\sigma_{B\,\mathrm{O}(n,n;\mathbb{F}_p)},\rho_{B\,\mathrm{O}(n,n;\mathbb{F}_p)}\right)$-module structures on $F = \sigma^3_{B\mathbb{F}_p^n}$.*

Proof. By Corollary 3.17(d), the trivialization defining the $\mathrm{O}\,(n,n;\mathbb{F}_p)$-theory in Corollary 5.28 (with underlying framed theory F) defines a $\left(\sigma_{B\,\mathrm{O}(n,n;\mathbb{F}_p)},\rho_{B\,\mathrm{O}(n,n;\mathbb{F}_p)}\right)$-module structure on $F = \sigma^3_{B\mathbb{F}_p^n}$. □

Appendix A. TQFT and category theory

The axioms for functorial quantum field theory originated in [Ati88, Seg04]. In this work, we will use TQFT to refer to *fully-extended functorial topological quantum field theories*, which first appeared in [BD95]. These were further studied in [Lur09]. Other references include [Fre93, Wal06, Tel16, SP14, Kap10].

A.1. **The Cobordism Hypothesis.** The Cobordism Hypothesis was formulated by Baez-Dolan in [BD95]. A detailed sketch of the proof was given in [Lur09], and an approach for a proof using factorization homology is in [AF17]. A proof in two dimensions is in [SP09]. See [Fre13, SP14] for other references.

Let $d \in \mathbb{Z}^{\geq 0}$. Given a symmetric monoidal $(\infty, d+1)$-category $\mathcal{T}$, which will be the target for our theories, write $\mathcal{T}^\sim$ for the groupoid obtained by discarding all non-invertible morphisms at all levels. We will assume $\mathcal{T}$ has duals.[14] We will also assume that $\Omega^{d+1}\mathcal{T} = \mathbf{k}$ for $\mathbf{k}$ an algebraically closed field of characteristic zero. Throughout the paper, we will write **Vect** for the $\mathbf{k}$-linear category of finite-dimensional vector spaces over $\mathbf{k}$.

Remark A.1. Besides having duals and satisfying $\Omega^{d+1}\mathcal{T} = \mathbf{k}$, we will occasionally assume that $\mathcal{T}$ satisfies an additional hypothesis: When we discuss twisted module structures, in the sense of [FMT22] (as yet another avatar of anomalous theories), we will assume that the target $\mathcal{T}$ is sufficiently additive to support the construction of π-finite spaces in the sense of [FHLT10, §8] and [FMT22, §A] (this is Hypothesis Q).

[14] Otherwise replace $\mathcal{T}$ with the subcategory consisting of the fully-dualizable objects of $\mathcal{T}$.

The (framed) Cobordism Hypothesis, [Lur09, Theorem 2.4.6], asserts that, since $\mathcal{T}$ has duals, the functor given by evaluation on the point is an equivalence:

$$\mathrm{Fun}^{\otimes}\left(\mathbf{Bord}_{d+1}^{\mathrm{fr}}, \mathcal{T}\right) \xrightarrow{\sim} \mathcal{T}^{\sim} \ .$$

Remark A.2. The category $\mathrm{Fun}^{\otimes}(\mathbf{Bord}, \mathcal{T})$ consists of symmetric monoidal functors with morphisms given by strong natural transformations (in the sense of [JFS17]). A priori this is an $(\infty, d+1)$-category, but the statement of the Cobordism Hypothesis is that it is actually an $(\infty, 0)$-category, and in particular it is equivalent to the $(\infty, 0)$-category $\mathcal{T}^{\sim}$.

We will occasionally use the variant of $\mathrm{Fun}^{\otimes}(\mathbf{Bord}, \mathcal{T})$ which has the same objects, however the 1-morphisms are lax natural transformations. We will write this as $\mathrm{Fun}^{\mathrm{lax}}(\mathbf{Bord}, \mathcal{T})$. See Remark A.11.

A.1.1. *The Cobordism Hypothesis for X-theories.* There is a refined version of the Cobordism Hypothesis obtained as follows. We follow [Lur09]. Let X be a topological space with real rank $d+1$ vector bundle ζ, and let M be a manifold of dimension $m \leq d+1$. An (X, ζ)-structure on M consists of

(i) a continuous map $f\colon M \to X$, and
(ii) an isomorphism of bundles $TM \oplus \mathbb{R}^{d+1-m} \simeq f^*\zeta$.

Write $\mathbf{Bord}_{d+1}^{(X,\zeta)}$ for the category of bordisms equipped with an (X, ζ)-structure. This is [Lur09, Definition 2.4.17]. We will refer to symmetric monoidal functors out of $\mathbf{Bord}_{d+1}^{(X,\zeta)}$ as $(d+1)$-dimensional (X, ζ)-TQFTs, or just (X, ζ)-theories. The Cobordism Hypothesis for (X, ζ)-theories is [Lur09, Theorem 2.4.18]. It is the equivalence

$$\mathrm{Fun}^{\otimes}\left(\mathbf{Bord}_{d+1}^{(X,\zeta)}, \mathcal{T}\right) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{O}(d+1)}\left(\widetilde{X}, \mathcal{T}^{\sim}\right) \ , \tag{A.3}$$

where $\widetilde{X}$ is the associated principal $\mathrm{O}(d+1)$-bundle of orthonormal frames in ζ, and the right-hand side denotes $\mathrm{O}(d+1)$-equivariant maps, for the natural action of $\mathrm{O}(d+1)$ on $\mathcal{T}^{\sim}$ [Lur09, Theorem 2.4.18].

Let G be a topological group with a continuous group homomorphism to the orthogonal group $\chi\colon G \to \mathrm{O}(d+1)$. Particularly important cases of (X, ζ)-structures are given by $X = BG$ and $\zeta = \left(\mathbb{R}^{d+1} \times EG\right)/G$ defined by χ. In this case, we sometimes call theories defined on $\mathbf{Bord}_{d+1}^{BG}$ *G-theories*.

Example A.4. Let G be trivial and $\chi = \iota$ be the inclusion. A (BG, ζ_ι)-structure is a framing.

Example A.5. Let $G = \mathrm{SO}(d+1)$ and $\chi = \iota$ be the inclusion. A $(B\,\mathrm{SO}(d+1), \zeta_\iota)$-structure is an orientation.

We will write $\mathbf{Bord}_{d+1}^{X}$ for the category of bordisms with (X, ζ)-structure with ζ trivial. E.g. $X = BG$ and $\chi\colon G \to \mathrm{O}(d+1)$ trivial. In spite of fr being removed from the notation, one should think of $\mathbf{Bord}_{d+1}^{X}$ as consisting of framed bordisms with maps to X. We will refer to symmetric monoidal functors $\mathbf{Bord}_{d+1}^{X} \to \mathcal{T}$ as *X-TQFTs* (or just X-theories). These should be thought of as families of framed TQFTs over X. In this case, the Cobordism Hypothesis (A.3) reduces to:

$$\mathrm{Fun}^{\otimes}\left(\mathbf{Bord}_{d+1}^{X}, \mathcal{T}\right) \xrightarrow{\sim} \mathrm{Hom}\left(X, \mathcal{T}^{\sim}\right) \ . \tag{A.6}$$

Example A.7. When $X = BG$ with $\chi = \mathrm{triv}\colon G \to \mathrm{O}(d+1)$ trivial, then the Cobordism Hypothesis reduces to [Lur09, Theorem 2.4.26]:

$$\mathrm{Fun}^{\otimes}\left(\mathbf{Bord}_{d+1}^{BG}, \mathcal{T}\right) \xrightarrow{\sim} \left(\mathcal{T}^{\sim}\right)^{hG}$$

where $(\mathcal{T})^{hG}$ denotes the homotopy fixed points of $\mathcal{T}$.

We will sometimes call symmetric monoidal functors $\mathbf{Bord}_{d+1}^{BG} \to \mathcal{T}$ *BG-theories*, or *G-theories*.

Endomorphisms of the trivial TQFT are equivalent to theories of one lower dimension. By the Cobordism Hypothesis this is equivalent to endomorphisms of the identity in the target being decategorification. Proposition A.9 is this result for X-theories.

Lemma A.8. *If $\mathcal{A}$ and $\mathcal{B}$ are monoidal $(\infty, d+1)$-categories, then*

$$\Omega \operatorname{Fun}^{\otimes}(\mathcal{A}, \mathcal{B}) \simeq \operatorname{Fun}^{\otimes}(\mathcal{A}, \Omega\mathcal{B}) \ .$$

Proof. The LHS consists of natural transformations:

$$\begin{array}{ccc} \mathcal{A} & \longrightarrow & \mathbf{1} \\ \downarrow & \nearrow\!\!\nearrow & \downarrow \\ \mathbf{1} & \longrightarrow & \mathcal{B} \end{array}$$

where $\mathbf{1}$ denotes the trivial monoidal category. The category $\Omega\mathcal{B}$ is the pullback of the diagram $\mathbf{1} \to \mathcal{B} \leftarrow \mathbf{1}$, so the result follows from the universal property. □

Proposition A.9. *If 1_X^{d+1} is the trivial functor $\mathbf{Bord}_{d+1}^X \to \mathcal{T}$, then*

$$\operatorname{Fun}^{\otimes}\left(\mathbf{Bord}_d^X, \Omega\mathcal{T}\right) \simeq \operatorname{End}\left(1_X^{d+1}\right) \ ,$$

where $\Omega\mathcal{T} \coloneqq \operatorname{End}_{\mathcal{T}}(1)$ denotes the looping of $\mathcal{T}$.

Proof. This follows from Lemma A.8 for $\mathcal{A} = \mathbf{Bord}_d^X$ and $\mathcal{B} = \mathcal{T}$. □

A.2. **Relative theories.** A symmetric monoidal functor

$$\alpha \colon \mathbf{Bord}_d^X \to \mathcal{T}$$

is a *once-categorified d-dimensional X-TQFT*, where $\mathcal{T}$ is the fixed target from the beginning of Section A.1. Let

$$1_X \colon \mathbf{Bord}_d^X \to \mathcal{T}$$

denote the trivial once-categorified d-dimensional X-TQFT. Recall the notion of a relative theory [FT14]. These are also called *twisted theories* [ST11, JFS17].

Definition A.10. A theory defined right-relative to α is a lax natural transformation (in the sense of [JFS17])

$$\alpha \to 1 \ .$$

A theory defined left-relative to α is a lax natural transformation

$$1 \to \alpha \ .$$

Remark A.11. Recall the definition of lax (resp. oplax) natural transformations from [JFS17]. Consider the arrow categories $\mathcal{T}^{\downarrow}$ and $\mathcal{T}^{\to}$, and the source and target functors $s, t \colon \mathcal{T}^{*} \to \mathcal{T}$ for $* = \downarrow, \to$. Following [JFS17], a lax (resp. oplax) natural transformation $\alpha \to 1$ is a functor

$$F_\alpha \colon \mathbf{Bord}_d^X \to \mathcal{T}^{\downarrow} \qquad \left(\text{resp. } F_\alpha \colon \mathbf{Bord}_d^X \to \mathcal{T}^{\to}\right) \tag{A.12}$$

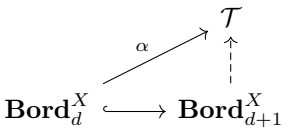
such that $s \circ F_\alpha = \alpha$, and $t \circ F_\alpha = 1$.

Throughout, we will use the lax version, as written in the above definition of relative theories. The reason we use the lax version, as noted in [JFS17, Example 7.3], that the *lax* natural transformations from the trivial theory to itself consist of theories of dimension lower ([JFS17, Theorem 7.4]) whereas the same is not true when lax is replaced with oplax. We need the analogous result for X-theories (Proposition A.9), in particular to establish a trivialized anomalous theory $1 \xrightarrow{\sim} \alpha \to 1$ as an honest theory of one dimension lower.

Also noted in [JFS17, Example 7.3], is the fact that oplax natural transformations are "elements" $F_\sigma(M) : 1 \to \alpha(M)$ for M a *closed* bordism of any codimension, which is for example the point of view taken in [FT14]. Oplax natural transformations are also used in [FT21]. Besides Proposition

A.9, and the results depending on it, the remaining constructions and facts in this paper hold for the oplax version, obtained by replacing all lax natural transformations with oplax ones, and replacing $\mathcal{T}^{\downarrow}$ with $\mathcal{T}^{\rightarrow}$.

Remark A.13. Often the once-categorified d-dimensional theory α extends to a $(d+1)$-dimensional theory:



In this case, the relative theory $\alpha \to 1$ is upgraded to what is called a *boundary theory*. Boundary theories are defined as functors out of the extended bordism category $\mathbf{Bord}_{d+1}^{X,\partial}$, described in [Lur09, §4.3]. See also [Ste24]. The connection with the notion defined here is made in [JFS17, Theorem 7.15].

A.3. **Invertible theories.** Let $\left|\mathbf{Bord}_{d+1}^X\right|$ denote the completion, i.e. the $(\infty,0)$-category obtained by manually inverting all the arrows, and let $\mathcal{T}^{\times}$ denote the underlying sub-groupoid, i.e. the $(\infty,0)$-category obtained by discarding all non-invertible objects and non-invertible morphisms at all levels.

A TQFT $\alpha\colon \mathbf{Bord}_{d+1}^X \to \mathcal{T}$ is *invertible* if it factors as:

(A.14)
$$\begin{array}{ccc} \mathbf{Bord}_{d+1}^X & \xrightarrow{\alpha} & \mathcal{T} \\ \downarrow & & \uparrow \\ \left|\mathbf{Bord}_{d+1}^X\right| & \xrightarrow{\widetilde{\alpha}} & \mathcal{T}^{\times} \end{array}$$

I.e. it assigns all objects and morphisms in the source to invertible objects and morphisms in the target.

Remark A.15. Note that (A.14) reduces the functor α to a map of spectra $\widetilde{\alpha}$, since both $\left|\mathbf{Bord}_{d+1}^X\right|$ and $\mathcal{T}^{\times}$ are infinite loop spaces on account of their symmetric monoidal structures [FH21]. This allows us to study it as a cohomology class (in the theory determined, via Brown representability, by the target) on the source.

Appendix B. Topological symmetry

B.1. **TQFTs associated to π-finite spaces.** Let X be a space (i.e. higher groupoid) which is (connected, pointed, and) π-finite[15]. There is a recipe for constructing a TQFT using X, which was introduced in [Kon88] and studied further in [Qui95, Tur10]. We will consider the fully local case introduced in [Fre94] and studied in [FHLT10, §3,8] and [FMT22, §A].

In the 1-dimensional case, the finite path-integral has mathematical rigorous foundations in the ∞-categorical setting [Har20]. Related work in a higher setting is the subject of an upcoming work of Claudia Scheimbauer and Tashi Walde.

B.1.1. *The summation map.* We will proceed heuristically, following [FHLT10, §3] and [FMT22, §A.2], to fix notation and describe expectations which will eventually be stated and assumed in Hypothesis Q.

Let $\mathbf{Fam}_{d+1}$ denote the category with objects finite $(d+1)$-groupoids, 1-morphisms given by correspondences of π-finite spaces, 2-morphisms given by correspondences of correspondences, and so on until level $(d+1)$. (Two $(d+1)$-morphisms are regarded as identical if they are equivalent.)

[15]This means X has finitely many nonzero homotopy groups, each of which is finite.

Let $\mathcal{T}$ be the symmetric monoidal target with duals, fixed in Section A.1. Let Y be an object of $\mathbf{Fam}_{d+1}$. A local system on Y valued in $\mathcal{T}$ is a functor $Y \to \mathcal{T}$. Write $\mathbf{Fam}_{d+1}(\mathcal{T})$ for the category of π-finite spaces equipped with a local system valued in $\mathcal{T}$, with morphisms given by correspondences equipped with morphisms between the pulled-back local systems.

For example, $\mathbf{Fam}_{d+1}\left(B^{d+1}\mathbf{k}^{\times}\right)$ has objects given by pairs (Y,τ), where τ is a cocycle

$$\tau\colon Y \to B^{d+1}\mathbf{k}^{\times}$$

representing a class in $H^{d+1}(Y,\mathbf{k}^{\times})$. A morphism is a correspondence:

$$\begin{array}{ccc} & (E,\mu) & \\ {\scriptstyle p_1}\swarrow & & \searrow{\scriptstyle p_2} \\ (Y_1,\tau_1) & & (Y_2,\tau_2) \end{array} \tag{B.1}$$

where $\mu\colon E \to B^{d}\mathbf{k}^{\times}$ satisfies

$$d\mu = \left(p_1^{*}\tau_1\right)^{-1}\cdot\left(p_2^{*}\tau_2\right)\ . \tag{B.2}$$

2-morphisms are correspondences between correspondences with a similar condition on the cocycles, and so on to define morphisms up to level $(d+1)$.

Recall we assumed $\Omega^{d+1}\mathcal{T} = \mathbf{k}$. Therefore there is a natural functor: $B^{d+1}\mathbf{k}^{\times} \to \mathcal{T}$ inducing a functor:

$$\mathbf{Fam}_{d+1}\left(B^{d+1}\mathbf{k}^{\times}\right) \to \mathbf{Fam}_{d+1}(\mathcal{T})\ . \tag{B.3}$$

We assume that the following holds.

Hypothesis Q. *For $\mathcal{T}$ appropriately additive*[16]*, there is a "quantization"*[17] *functor*

$$\mathrm{Sum}_{d+1}\colon \mathbf{Fam}_{d+1}\left(B^{d+1}\mathbf{k}^{\times}\right) \to \mathcal{T} \tag{B.4}$$

such that there is an invertible natural transformation between

$$(X,\tau) \mapsto \mathrm{Hom}_{\mathcal{T}}\left(1, \mathrm{Sum}_{d+1}(X,\tau)\right)$$

and

$$(X,\tau) \mapsto \mathrm{Hom}_{\tau}\left(\mathrm{hofib}(\tau), \Omega\mathcal{T}\right)$$

viewed as functors out of $\mathbf{Fam}_{d+1}\left(B^{d+1}\mathbf{k}^{\times}\right)$.

Here $\mathrm{Hom}_{\tau}\left(\mathrm{hofib}(\tau),\Omega\mathcal{T}\right)$ denotes the functors $\mathrm{hofib}(\tau) \to \Omega\mathcal{T}^{\sim}$ whose restriction to each fiber $B^{d}\mathbf{k}^{\times}$ is the canonical inclusion of scalars.

Remark B.5. Unpacking the existence of the natural transformation in Hypothesis Q, we see that it ensures that we have equivalences

$$\mathrm{Hom}_{\mathcal{T}}\left(1, \mathrm{Sum}_{d+1}(X,\tau)\right) \simeq \mathrm{Hom}_{\tau}\left(\mathrm{hofib}(\tau), \Omega\mathcal{T}\right) \tag{B.6}$$

for all objects (X,τ) of $\mathbf{Fam}_{d+1}\left(B^{d+1}\mathbf{k}^{\times}\right)$, and given a morphism from (X_1,τ_1) to (X_2,τ_2) in $\mathbf{Fam}_{d+1}\left(B^{d+1}\mathbf{k}^{\times}\right)$, we have a commuting diagram:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{T}}\left(1, \mathrm{Sum}_{d+1}(X_1,\tau_1)\right) & \simeq & \mathrm{Hom}_{\tau_1}\left(\mathrm{hofib}(\tau_1), \Omega\mathcal{T}\right) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathcal{T}}\left(1, \mathrm{Sum}_{d+1}(X_2,\tau_2)\right) & \simeq & \mathrm{Hom}_{\tau_2}\left(\mathrm{hofib}(\tau_2), \Omega\mathcal{T}\right) \end{array} \tag{B.7}$$

[16] As is explained in [FHLT10, §3], for X an ordinary groupoid, then $\mathcal{T}$ must be additive in the sense that the colimit $\varinjlim_{x\in X}\tau(x)$ in $\mathcal{T}$ exists, and agrees with the limit $\varprojlim_{x\in X}\tau(x)$, where we are regarding τ as defining a $\mathcal{T}$-valued local system on X by (B.3).

[17] See [FMT22, Remark A.7.1] where it is explained how $\mathrm{Sum}_{d+1}(X)$ (and therefore σ_X^{d+1}) can be obtained by integrating over fluctuating fields.

This generalizes the classical fact about modules over the group algebra and G-representations, as in Example B.9 (ii).

Remark B.8. In [FHLT10, §8.2] the map Sum_n is constructed at the level of objects and shown to be a functor up to 2-morphisms. In particular, the cases in Example B.9 are worked out in [FHLT10, §8.1,8.3].

Example B.9. (i) Let $\mathcal{T} = \mathbf{Vect}$. Then $\mathrm{Sum}_1(X) = \mathbf{k}(\pi_0 X)$ is the vector space of $\mathbf{k}$-valued functions on $\pi_0 X$. Hypothesis Q is satisfied: the natural transformation (B.6) is given by the following equivalences, whose naturality is straightforward to check:

$$\mathrm{Hom}_{\mathbf{Vect}}(1, \mathrm{Sum}_1(X)) \simeq \mathrm{Hom}(X, \mathbf{k}) \simeq \mathrm{Hom}(\pi_0 X, \mathbf{k}) \ .$$

(ii) Set $X = BG$ for a finite group G, and let $\mathcal{T} = \mathbf{Alg}$ be the Morita 2-category of algebras. Let $\mathrm{Sum}_2(BG) = \mathbf{k}[G]$ be the group algebra. Hypothesis Q is satisfied: the natural transformation (B.6) is given by the following equivalences, whose naturality is straightforward to check:

$$\mathrm{Hom}_{\mathbf{Alg}}(1, \mathrm{Sum}_2(BG)) = \mathbf{k}[G]\text{-}\mathbf{mod} \simeq \mathbf{Rep}(G) \simeq \mathrm{Hom}(BG, \mathbf{Vect}) \ .$$

We can equip BG with a 2-cocycle μ (i.e. a group 2-cocycle) which twists the convolution structure on the group algebra, resulting in $\mathrm{Sum}_2(BG, \tau)$.

(iii) Set $X = BG$. Let $d = 2$, and take $\mathcal{T}$ to be the Morita 3-category of fusion categories. Then $\mathrm{Sum}_3(BG)$ is $\mathbf{Vect}[G]$, the fusion category of vector bundles on G with convolution. We can equip BG with a 3-cocycle α which twists the fusion structure on $\mathbf{Vect}[G]$, yielding $\sigma^3_{BG,\alpha}(*)$. This is [FMT22, Example A.65].

(iv) Set $X = B^2G$. Let $d = 3$ and take $\mathcal{T}$ to be the Morita 4-category of braided monoidal categories, $\mathbf{BrFus}$ (see Section 4.1). Consider a cocycle $\tau\colon B^2G \to B^4\mathbf{k}^\times$. It is a theorem of Eilenberg-Mac Lane [EL54, Theorem 26.1] that cohomology classes in $H^4\left(B^2G, \mathbf{k}^\times\right)$ correspond to quadratic forms $G \to \mathbf{k}^\times$. Then $\mathrm{Sum}_4\left(B^2G\right)$ is $\mathbf{Vect}[G]$ with convolution, and with associator and braiding (4.19) given by an abelian 3-cocycle (τ_0, b) representing $[\tau]$ (see Section 4.3.2).

(v) Let $d = 3$ and $\mathcal{T}$ be a 3-category of monoidal 2-categories (e.g. $2\mathbf{Fus}$ [DR18]). For any finite group G, the fusion 2-category $\mathrm{Sum}_4(BG)$ is the collection of G-graded 2-vector spaces [DR18, Construction 2.1.13].

As far as the author is aware, dualizability in the Morita 4-category of monoidal 2-categories has not been extensively studied. However this fusion 2-category is expected define a fully extended 4-dimensional TQFT [DR18]. See Section 5.8 where we discuss this example in more detail.

B.1.2. *TQFTs from groupoids.* The upshot of assuming Hypothesis Q is that, for a fixed object (X, τ) of $\mathbf{Fam}_{d+1}\left(B^{d+1}\mathbf{k}^\times\right)$, (B.4) can be composed with the mapping space functor[18] to obtain the theory

$$\sigma^{d+1}_{X,\tau}\colon \mathbf{Bord}^{\mathrm{fr}}_{d+1} \xrightarrow{\pi_{\leq d+1}\mathrm{Map}(-,X)} \mathbf{Fam}_{d+1}\left(B^{d+1}\mathbf{k}^\times\right) \xrightarrow{\mathrm{Sum}_{d+1}} \mathcal{T} \ .$$

Remark B.10. In [FHLT10, §3,8], the theories σ^{d+1}_X are studied for $\mathcal{T} = \mathbf{Alg}[d]$ the Morita $(d+1)$-category of "d-algebras", discussed in [FHLT10, §7].

Remark B.11. As is remarked in [FMT22, §A] and [FHLT10, §3,8], $\sigma^{d+1}_{X,\tau}$ can be upgraded to an oriented theory, and if τ is trivial then it can even be upgraded to an unoriented theory. We will work with the underlying framed theories in this paper.

[18] Note that (B.3) allows us to construct a $\mathcal{T}$-valued local system from the cocycle τ.

Proposition B.12 ([FMT22]). *Assuming Hypothesis Q, a morphism from (X_1, τ_1) to (X_2, τ_2) in $\mathbf{Fam}_{d+1}\left(B^{d+1}\mathbf{k}^\times\right)$ (i.e. correspondence as in (B.1)) induces a bimodule (i.e. domain wall)*

$$\sigma^{d+1}_{X_1,\tau_1} \to \sigma^{d+1}_{X_2,\tau_2} \ .$$

Proof. By the Cobordism Hypothesis, the functor Sum_{d+1} in Hypothesis Q provides a functor:

$$\begin{aligned} \mathbf{Fam}_{d+1}\left(B^{d+1}\mathbf{k}^\times\right) &\longrightarrow \mathrm{Fun}^\otimes\left(\mathbf{Bord}^{\mathrm{fr}}_{d+1}, \mathcal{T}\right) \\ (X,\tau) &\longmapsto \sigma^{d+1}_{X,\tau} \end{aligned}$$

Proposition B.12 follows from the fact that the correspondence is a morphism in the category $\mathbf{Fam}_{d+1}\left(B^{d+1}\mathbf{k}^\times\right)$. □

Remark B.13. Proposition B.12 can be shown directly (i.e. without the Cobordism Hypothesis) since a correspondence of spaces $X \leftarrow C \to Y$ defines a correspondences of mapping spaces $\mathrm{Map}(M, X) \leftarrow \mathrm{Map}(M, C) \to \mathrm{Map}(M, Y)$, for any bordism M. This is the perspective taken in [FMT22].

Recall the motivation in Remark B.5 for the existence of the natural transformation in Hypothesis Q. Proposition B.14 writes this in terms of the theories σ^{d+1}_X.

Proposition B.14. *Assuming Hypothesis Q, every boundary theory for $\sigma^{d+1}_{X,\tau}$ is classified by a symmetric monoidal functor:*

$$\mathbf{Bord}_d^{\mathrm{hofib}(\tau)} \to \Omega\mathcal{T}$$

whose classifying map lies in $\mathrm{Hom}_\tau\left(\mathrm{hofib}(\tau), \Omega\mathcal{T}\right)$. If τ is trivial, then the boundary theories are classified by symmetric monoidal functors:

$$\mathbf{Bord}_d^X \to \Omega\mathcal{T} \ .$$

Proof. By [JFS17, Theorem 7.15], the boundary theories for $\sigma^{d+1}_{X,\tau}$ are classified by

$$\mathrm{Hom}_{\mathcal{T}}\left(1, \sigma^d_{X,\tau}(*)\right)$$

which is the LHS of (B.6). Similarly, by the Cobordism Hypothesis (A.6), the symmetric monoidal functors from $\mathbf{Bord}_d^{\mathrm{hofib}(\tau)}$ to $\Omega\mathcal{T}$ are classified by $\mathrm{Hom}\left(\mathrm{hofib}(\tau), \Omega\mathcal{T}\right)$, and the RHS of (B.6) is the subspace Hom_τ. If τ is trivial, then a trivialization identifies $\mathrm{hofib}(\tau) \simeq X \times B^d\mathbf{k}^\times$, so the RHS of (B.6) is equivalent to functors from X to $\Omega\mathcal{T}^\sim$. □

B.2. **Module structures.** We summarize the material used in our construction from [FMT22]. See [FMT22, §3] for a more detailed discussion of these definitions.

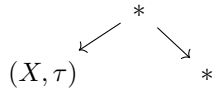
Let

$$\sigma\colon \mathbf{Bord}^{\mathrm{fr}}_{d+1} \to \mathcal{T}$$

be a $(d+1)$-dimensional TQFT valued in the fixed target $\mathcal{T}$ from the beginning of Section A.1. Recall the notion of a boundary theory from Remark A.13. Recall the following definition from [FMT22].

Definition B.15. A *d-dimensional quiche* is a pair (σ, ρ) in which ρ is a right topological boundary theory (or *right σ-module*), which we will write as $\rho\colon \sigma \to 1$.

Remark B.16. All of the quiches considered in this paper will be of the form $(\sigma_{X,\tau}, \rho_{X,\tau})$, where $\sigma_{X,\tau}$ is the theory associated to (X, τ) as in Section B.1, and the (right) boundary theory $\rho_{X,\tau}$ is the natural transformation induced by the correspondence diagram:



as in Proposition B.12. Given a pointed space X, we will always write $\rho_{X,\tau}$ for this boundary theory.

Recall, from [JFS17, Corollary 7.7], there is an equivalence of $(\infty, d+1)$-categories between the following.

(1) The $(\infty, d+1)$-category $\mathrm{Fun}^{\mathrm{lax}}\left(\mathbf{Bord}_{d+1}^{\mathrm{fr}}, \mathcal{T}\right)$ of framed TQFTs with 1-morphisms given by relative[19] field theories between them, and k-morphisms given by what Johnson-Freyd-Scheimbauer call k-times-twisted field theories.
(2) The $(\infty, d+1)$-subcategory of $\mathcal{T}$ consisting of fully dualizable objects, and with morphisms which are $(d+1)$-times left-adjunctible.

Given any quiche (σ, ρ), the boundary theory ρ is equivalent to a relative theory $\tau_{\leq d}\sigma \to 1$ by [JFS17, Theorem 7.15], and is classified by a 1-morphism

$$|\rho| : \sigma(*) \to 1$$

in $\mathcal{T}$.

Definition B.17. A morphism between quiches $(\sigma, \rho) \to (\sigma', \rho')$ is a 1-morphism $\delta\colon \sigma(*) \to \sigma'(*)$ in $\mathcal{T}$ along with an invertible 2-morphism $s\colon |\rho| \Rightarrow |\rho'| \circ \delta$.

A quiche is an abstract symmetry datum, in the same sense as an algebra. The following definition, from [FMT22], is the analogue of a module, i.e. a realization of the quiche as symmetries of a given theory.

Definition B.18. Let (σ, ρ) be a d-dimensional quiche, and let F be a d-dimensional TQFT. A (σ, ρ)-module structure on F is the pair (F_σ, θ) where F_σ is a (left) boundary theory $F_\sigma\colon 1 \to \sigma$ which is equipped with an isomorphism of d-dimensional theories:

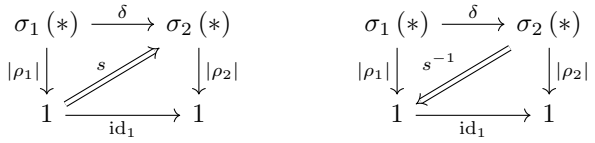
$$\theta\colon \rho \otimes_\sigma F_\sigma \xrightarrow{\sim} F\ .$$

B.3. **Reduction of topological symmetry.** Let X be a (pointed, connected) π-finite space, and consider a cocycle $c\colon X \to B^{d+1}\mathbf{k}^\times$. Assuming Hypothesis Q, and given a $\left(\sigma_{X,c}^{d+1}, \rho_{X,c}\right)$-module structure on a theory F, we might wonder what extra data is needed to "reduce" this to a $\left(\sigma_X^{d+1}, \rho_X\right)$-module structure on F. Theorem B.19, which holds more generally, will tell us that a trivialization of c is sufficient to perform such an operation, which we will define to be a *reduction* of topological symmetry in Definition B.24.

Recall the analogy in [FMT22] between modules over an algebra (or linear representations of a Lie group) and field theories. Under this analogy, Theorem B.19 is the analogue of the fact that a bimodule induces a functor between the categories of modules.

Theorem B.19. *Assume Hypothesis Q. Given a (σ_1, ρ_1)-module structure on a d-dimensional theory F, then a morphism of quiches (Definition B.17) from (σ_1, ρ_1) to (σ_2, ρ_2) canonically defines a (σ_2, ρ_2)-module structure on F.*

Proof. Write the given (σ_1, ρ_1)-module structure on F as (F_{σ_1}, θ_1) (as in Section B.2). A morphism of quiches (Definition B.17) consists of an invertible 2-morphism:



[19] Recall from Remark A.11 we are using lax natural transformations/relative theories in this paper.

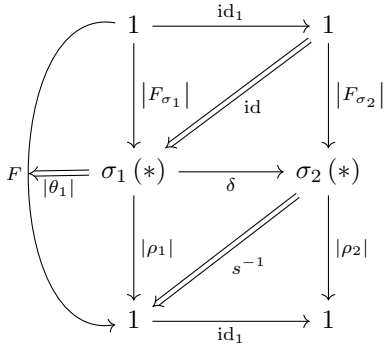
The morphism δ induces a (σ_2, σ_1)-bimodule, i.e. a domain wall $D\colon \sigma_1 \to \sigma_2$. Define the (σ_2, ρ_2)-module structure (F_{σ_2}, θ_2) as follows. Define the boundary theory F_{σ_2} to be

$$F_{\sigma_2} = D \otimes_{\sigma_1} F_{\sigma_1} \ .$$

Define the equivalence

$$\theta_2\colon \rho_2 \otimes_{\sigma_2} F_{\sigma_2} = \rho_2 \otimes_{\sigma_2} D \otimes_{\sigma_1} F_{\sigma_1} \xrightarrow{\sim} F$$

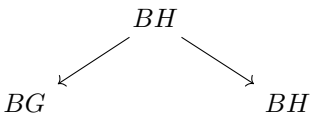
to be induced by the following composition:



where $|\theta_1|$ is the 2-morphism in $\mathcal{T}$ classifying the equivalence:

$$\theta_1\colon \rho_1 \otimes_{\sigma_1} F_{\sigma_1} \xrightarrow{\sim} F \ .$$ □

Example B.20. The inclusion of a subgroup $H \subset G$ defines a correspondence



which induces a morphism of quiches

$$\left(\sigma_{BG}^{d+1}, \rho_{BG}\right) \to \left(\sigma_{BH}^{d+1}, \rho_{BH}\right) \ .$$

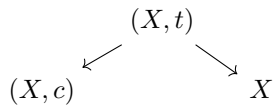
If a framed d-dimensional TQFT F has a $\left(\sigma_{BG}^{d+1}, \rho_{BG}\right)$-module structure, then the induced $\left(\sigma_{BH}^{d+1}, \rho_{BH}\right)$-module structure from Theorem B.19 is the restriction of the original module structure along the inclusion of H. In Example B.9 (ii) this is literally the restriction of a G-representation to an H-representation.

Now we return to X a π-finite space with $c\colon X \to B^{d+1}\mathbf{k}^\times$.

Proposition B.21. *A trivialization of the class c induces a $\left(\sigma_X^{d+1}, \rho_X\right)$-module structure on any theory with a $\left(\sigma_{X,c}^{d+1}, \rho_{X,c}\right)$-module structure.*

Proof. A trivialization t of c determines a correspondence as in (B.1):

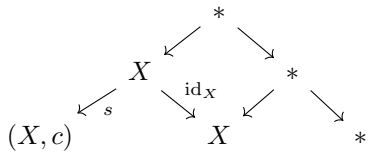
(B.22)



By Proposition B.12, this induces a domain wall D from $\sigma_{X,c}^{d+1}$ to σ_X^{d+1}. This tautologically induces a morphism of quiches (Definition B.17) from $\left(\sigma_{X,c}^{d+1}, \rho_{X,c}\right)$ to

$$\left(\sigma_X, \rho_{X,c} \otimes_{\sigma_{X,c}^{d+1}} D\right) \ .$$

But now notice that $\rho_{X,c} \otimes_{\sigma_{X,c}^{d+1}} D$ is induced by the following composition of correspondences:

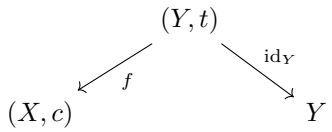


But this is just the pointing of (X, c), i.e. it induces $\rho_{X,c}$. Then Theorem B.19 implies the result. □

We can pull c back along a map $f\colon Y \to X$ so, even if it is not trivializable on X, it may be on Y.

Proposition B.23. *A trivialization of f^*c induces a $\left(\sigma_Y^{d+1}, \rho_Y\right)$-module structure on any theory with a $\left(\sigma_{X,c}^{d+1}, \rho_{X,c}\right)$-module structure.*

Proof. A trivialization t of f^*c determines a correspondence



and the rest of the proof is the same as that of Proposition B.21. □

Definition B.24. Let $f\colon Y \to X$ be a map between π-finite spaces. Given a trivialization of $f^*c\colon Y \to B^{d+1}\mathbf{k}^\times$, we will refer to the induced module structure from Proposition B.23 as a *reduction to* Y of the original module structure.

Example B.25. Let $\widetilde{X} = B\widetilde{G} \to BG = X$ for a group extension $\widetilde{G} \to G$. I.e. $d = 1$, and the class c classifies this central extension of G as usual. Theories with a $\left(\sigma_{B\widetilde{G}}, \rho_{B\widetilde{G}}\right)$-module structure are the same as representations of $\widetilde{G}$, i.e. projective representations of G with projectivity cocycle c. Proposition B.23 then says that splittings of $\widetilde{G}$ over $H \to G$ determine reductions of the linear representations of $\widetilde{G}$ to linear representation of H, as usual. See Example 3.9 and [Fre23, §1].

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