PROJECTIVE SYMMETRIES OF THREE-DIMENSIONAL TQFTS

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Abstract. Quantum field theory has various projective characteristics which are captured by what are called anomalies. This paper explores this idea in the context of fully-extended three-dimensional topological quantum field theories (TQFTs).

Given a three-dimensional TQFT (valued in the Morita 3-category of fusion categories), the anomaly identified herein is an obstruction to gauging a naturally occurring group of symmetries (an orthogonal group when the Drinfeld center is pointed). In other words, the classical symmetry group almost acts: There is a lack of coherence at the top level. This lack of coherence is captured by a "higher (central) extension" of the symmetry group, obtained via a modification of the obstruction theory of Etingof-Nikshych-Ostrik-Meir. This extension tautologically acts on the given TQFT/fusion category, and this precisely classifies a *projective* (equivalently *anomalous*) TQFT. We explain the sense in which this is an analogue of the classical spin representation. This is an instance of a phenomenon emphasized by Freed: Quantum theory is projective.

We also establish, under some assumptions, a general relationship between the language of projectivity/anomalies and the language of topological symmetries. We also identify a universal anomaly associated with any theory which is appropriately "simple".

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1. Introduction

Quantum field theory is fundamentally projective [Fre23, §3]. For instance, the correlation functions of a given QFT are unchanged by tensoring with an invertible theory. One example of this phenomenon, in quantum mechanics, is the well-known fact that the pure states form the *projectivization* of the Hilbert space of all states. From the perspective of symmetries, this projectivity appears when one asks if a certain group acts coherently on the Hilbert space: The (symplectic) group of symmetries of the phase space will only act projectively, i.e. only on the pure states.

This projectivity is captured by what is called an *anomaly*. Though anomalies often assume various mathematical guises, we will work with anomalies which are captured by invertible once-categorified theories. Anomalous theories are then theories defined relative to these invertible theories [Fre23].

1.1. Main results.

The main proposal of this work is that the Postnikov tower of the higher automorphism groupoid of the value of a theory on the point is a natural source of anomalies. We first develop a general framework for this (Theorem A), and then specialize to three dimensions, where it gives an anomaly of Turaev-Viro theories (Theorem B), refined to 3-group (i.e. twice-categorified group) analogues of Pin and Spin for pointed Drinfeld centers (Theorem C), and a projective action of braided autoequivalences on Witten-Reshetikhin-Turaev theories (Theorem D).

Following suggestions of Freed [Fre23], we define the equivalent notion of a *projective* TQFT. Fix a symmetric monoidal $(\infty, d+1)$-category $\mathcal{T}$ with duals, so that d-dimensional theories are valued in $\Omega\mathcal{T} = \mathrm{End}_{\mathcal{T}}(1)$. Unless otherwise specified, all categories are $\mathbf{k}$-linear for $\mathbf{k}$ algebraically

closed of characteristic zero, and we assume $\Omega^{d+1}\mathcal{T} \simeq \mathbf{k}$.[1] We define a projectivization $\mathbb{P}\Omega\mathcal{T}$, consisting of arrows from an invertible object to the unit in the lax arrow category $\mathcal{T}^{\downarrow}$ of [JFS17] (Definition 2.3); a projective theory is a theory valued in $\mathbb{P}\Omega\mathcal{T}$, and the source of the arrow is its *projectivity*. Since $\mathbb{P}$ commutes with iterated looping, *projective* 3-dimensional TQFTs give rise to *projective* representations of mapping class groups of surfaces (Section 1.2).

We also define what it means for a framed theory to have an *anomaly* as an X-theory (Definition 3.4): it is well-defined on $\mathbf{Bord}_d^{\mathrm{fr}}$, but only as a theory relative to an invertible once-categorified theory on $\mathbf{Bord}_d^X$ (framed bordisms with maps to X; Section A.1.1). This includes framing anomalies (Example 3.6) as well as the 't Hooft anomalies studied in this paper. Under a hypothesis on $\mathcal{T}$, anomalies can be rephrased in the language of topological symmetry [FMT22]. This hypothesis, Hypothesis Q, asserts that $\mathcal{T}$ admits a "quantization" functor sending a π-finite space X with a cocycle c to the twisted finite gauge theory $\sigma_{X,c}$, whose morphisms from the unit are the c-twisted $\Omega\mathcal{T}$-valued local systems on X. This rephrasing generalizes the fact that projective representations of a group G with cocycle c are the same as modules over the c-twisted group algebra.

Theorem A. *(i) There is a projectivization $\mathbb{P}\Omega\mathcal{T}$ (Definition 2.3), which depends functorially on $\mathcal{T}$ (with respect to symmetric monoidal equivalences) and commutes with iterated looping (Theorem 2.4).*

(ii) Projective TQFTs $\mathbf{Bord}_d^X \to \mathbb{P}\Omega\mathcal{T}$ with projectivity α are the same as anomalous TQFTs $\alpha \to 1$ (Proposition 3.5).

(iii) Let X be a π-finite space with a cocycle $c\colon X \to B^{d+1}\mathbf{k}^\times$. Assuming Hypothesis Q, an X-anomaly $\alpha_c \to 1_X$ of a framed theory F (Definition 3.4; here 1_X is the trivial once-categorified X-theory, Section A.2) determines a module structure on F over the $(d+1)$-dimensional finite gauge theory $\sigma_{X,c}$ twisted by c, in the sense of [FMT22] *(Theorem 3.12).*

The anomalies studied in this paper arise as follows. If the value $\mathcal{S}$ of a theory on the point is a Schur object (Definition 3.14), then $B\operatorname{Aut}(\mathcal{S})$ is a $B^d\mathbf{k}^\times$-bundle over its truncation $X = \pi_{\le d-1}B\operatorname{Aut}(\mathcal{S})$, classified by its top k-invariant $c\colon X \to B^{d+1}\mathbf{k}^\times$. The theory then has an X-anomaly α_c as in Theorem A, which we call the *universal anomaly* of $\mathcal{S}$ (Corollary 3.17). Theorem B below is an instance of this, and Theorem C is obtained from it by pulling back.

From now on we restrict our attention to anomalies of 3-dimensional theories. The anomaly theories will be valued in the (even higher [JFS17]) Morita 4-category[2] of braided tensor categories **BrTens**, or rather its subcategory **BrFus** consisting of rigid, finite, and semisimple tensor categories, which are fully-dualizable [BJS21].

One justification for this setting is that the group $\pi_0\mathbf{BrFus}^\times$ of equivalence classes of invertible objects[3] is the Witt group of braided fusion categories [BJSS21, Theorem 4.2]. This means there are more anomaly theories valued in **BrFus** than in, say, the category of vector spaces or the Morita categories of algebras or of fusion categories, all of which have trivial groups of invertible objects.

Every fusion category $\mathcal{C}$ is fully-dualizable [DSPS20], so it defines a fully-extended framed 3-dimensional TQFT F, sometimes called the Turaev-Viro theory associated with $\mathcal{C}$. This theory is

[1] Notation which is not completely standard is described informally below, and introduced precisely in the body of the paper: $\mathbf{Bord}_d^{(X,\zeta)}$ and $\mathcal{T}^{\sim}$ in Section A.1, the arrow category $\mathcal{T}^{\downarrow}$ in Section 2, $\mathbb{P}$ in Definition 2.3, quiches and Hypothesis Q in Appendix B, 3Lip, 3Pin, 3Spin, and the norm $N_{(A,q)}$ in Section 4.5, and polarized metric groups in Definition 4.30.

[2] We will generally use n-category to denote (∞,n)-category in the complete Segal space formalism, and otherwise we specify that we are considering a discrete n-category. We will sometimes still write (∞,n)-category to emphasize the ∞-categorical nature of the target categories. In particular, we work with the even higher Morita category of $\mathbb{E}_2$-algebras in (a suitable subcategory of) the symmetric monoidal 2-category of presentable categories. See Remark 4.1 for the relationship with the abelian category theory literature.

[3] For a higher category $\mathcal{T}$ we write $\mathcal{T}^\times$ for its invertible part, the ∞-groupoid of invertible objects and invertible morphisms at all levels (Section A.3).

well-defined as a framed theory; its anomaly only appears when we attempt to make it equivariant with respect to a certain natural group, or in some cases 2-group, of symmetries. In physical terms, we are identifying an obstruction to gauging the symmetry. The relevant symmetries are the braided autoequivalences of the Drinfeld center: the 3-type $B\operatorname{Aut}_{\mathbf{Fus}}(\mathcal{C})$ is a $B^3\mathbf{k}^\times$-bundle over $B\operatorname{Aut}_{\mathbf{EqBr}}(\mathcal{Z}(\mathcal{C}))$, classified by its k-invariant k_3 [ENO10].

Theorem B. *The framed TQFT F has a $B\operatorname{Aut}_{\mathbf{EqBr}}(\mathcal{Z}(\mathcal{C}))$-anomaly, given by the anomaly theory*

$$\alpha_{k_3}\colon \mathbf{Bord}_3^{B\operatorname{Aut}_{\mathbf{EqBr}}(\mathcal{Z}(\mathcal{C}))} \to \mathbf{BrFus}$$

classified by k_3. If $\mathcal{Z}(\mathcal{C})$ is pointed and $\mathcal{C} \not\simeq \mathbf{Vect}$, then the class k_3 is nontrivial (Theorem 5.10).

In other words, the automorphisms of $\mathcal{C}$ in **Fus** are determined by the braided autoequivalences of $\mathcal{Z}(\mathcal{C})$ up to a lack of coherence at the top level, and α_{k_3} is precisely this lack of coherence.

If $\mathcal{Z}(\mathcal{C})$ is pointed, it is classified by a metric group (A,q), i.e. a finite abelian group with a nondegenerate quadratic form, and A has a Lagrangian subgroup L (Proposition 4.29); a choice of L makes (A,q,L) a polarized metric group. The group of components of $\operatorname{Aut}_{\mathbf{EqBr}}(\mathcal{Z}(\mathcal{C}))$ is the orthogonal group $\mathrm{O}(A,q)$. If moreover $|A|$ is odd, this 2-group splits canonically, and k_3 pulls back to a class $O_4(A,q) \in H^4(B\,\mathrm{O}(A,q),\mathbf{k}^\times)$, which we refine to central[4] 3-group extensions $3\mathrm{Pin}(A,q)$ and $3\mathrm{Spin}(A,q)$ of $\mathrm{O}(A,q)$ and $\mathrm{SO}(A,q)$ by higher scalars (Section 4.5). These are classified by classes $c(3\mathrm{Pin}) \in H^4(B\,\mathrm{O}(A,q);\mu_{l^4})$ lifting $O_4(A,q)$, and its restriction $c(3\mathrm{Spin}) \in H^4(B\,\mathrm{SO}(A,q);\mu_{l^4})$, where $l = |L|$ (so $l^2 = |A|$) and $\mu_{l^4} \subset \mathbf{k}^\times$ is the group of l^4-th roots of unity.

Theorem C. *(i) If $\mathcal{Z}(\mathcal{C})$ is pointed with $|A|$ odd, then F has an $\mathrm{O}(A,q)$-anomaly $\alpha_{c(3\mathrm{Pin})} \simeq \alpha_{O_4}$, the pullback of α_{k_3} along the canonical splitting. The same holds for $\mathrm{SO}(A,q)$ and $3\mathrm{Spin}(A,q)$ (Theorem 5.17 and Corollary 5.21).*

(ii) For $\mathcal{C} = \mathbf{Vect}\left[\mathbb{F}_p^n\right]$ with p odd this anomaly is trivializable, and there is a canonical $H^3(B\,\mathrm{O}(n,n;\mathbb{F}_p),\mathbf{k}^\times)$-torsor of linear (i.e. non-projective) $\mathrm{O}(n,n;\mathbb{F}_p)$-TQFTs $\mathbf{Bord}_3^{B\,\mathrm{O}(n,n;\mathbb{F}_p)} \to \mathbf{Fus}$ with underlying framed theory F (Corollary 5.29).

Section 4.7 develops a detailed analogy with the classical spin representation (see also Table 1). One can also compare with the Weil representation of the metaplectic group (Remark 4.58), under which part (ii) corresponds to the splitting of the Weil representation over a finite field [GH09, GH11].

Theorems B and C concern the Turaev-Viro theory F of $\mathcal{C}$, which is the Witten-Reshetikhin-Turaev (WRT) theory [Wit89, RT91, Tur94] of the Drinfeld center $\mathcal{Z}(\mathcal{C})$ [BK10, Bal10a, Bal10b]. Any nondegenerate braided fusion category $\mathcal{A}$ defines a projective theory

$$\mathbf{WRT}_{\mathcal{A}}\colon \mathbf{Bord}_3^{\mathrm{fr}} \to \mathbb{P}\mathbf{Fus}\ ,$$

sending the point to $\mathcal{A}$ as a module over itself [Fre12, Haï24]. Since $\mathcal{A}$ is invertible in **BrFus** [BJSS21], its projectivity is the restriction to $\mathbf{Bord}_3^{\mathrm{fr}}$ of the invertible theory $\mathbf{CY}_{\mathcal{A}}\colon \mathbf{Bord}_4^{\mathrm{fr}} \to \mathbf{BrFus}$ sending the point to $\mathcal{A}$, a fully-extended framed version of the Crane–Yetter TQFT. For $\mathcal{A} = \mathcal{Z}(\mathcal{C})$, the module $\mathcal{C}$ trivializes $\mathbf{CY}_{\mathcal{Z}(\mathcal{C})}$, and the resulting linear theory sends the point to

$$\mathcal{C} \otimes_{\mathcal{Z}(\mathcal{C})} \mathcal{Z}(\mathcal{C}) \simeq \mathcal{C} \in \mathbf{Fus}\ ,$$

i.e. it is F. When $\mathcal{A}$ is in a nontrivial Witt class, on the other hand, $\mathbf{WRT}_{\mathcal{A}}$ does not extend to the point as an oriented (or even framed) theory valued in **Fus**. This illustrates one feature of the projective target category $\mathbb{P}$**Fus**: it allows for WRT theories (with a nontrivial framing anomaly) to be extended to the point. For general $\mathcal{A}$, braided autoequivalences act projectively on $\mathbf{WRT}_{\mathcal{A}}$:

Theorem D. *Let $\mathcal{A}$ be a nondegenerate braided fusion category, and consider a monoidal functor $\Phi\colon G \to \operatorname{Aut}_{\mathbf{EqBr}}(\mathcal{A})$ for a group G. There is an anomalous G-TQFT*

$$\mathbf{WRT}_{\mathcal{A}}^{G}\colon \mathbf{Bord}_3^{BG} \to \mathbb{P}\mathbf{Fus}$$

[4] See Remark 4.41.

which agrees with $\mathbf{WRT}_{\mathcal{A}}$ *upon restriction to trivial* G*-bundles. The anomaly theory of* $\mathbf{WRT}_{\mathcal{A}}^{G}$ *is the restriction to* $\mathbf{Bord}_3^{BG}$ *of a theory* $\mathbf{CY}_{\mathcal{A}}^{G}\colon \mathbf{Bord}_4^{BG} \to \mathbf{BrFus}$*, which agrees with* $\mathbf{CY}_{\mathcal{A}}$ *upon restriction to trivial* G*-bundles (Theorem 5.4).*

Finally, Proposition 4.45 and Corollary 5.28 show that every nontrivial class $[\pi] \in H^4(BG, \mathbf{k}^\times)$ of a finite group G arises in this way: it is $(B\Phi)^* k_3$ for some nondegenerate braided fusion category $\mathcal{B}$ and monoidal functor $\Phi\colon G \to \mathrm{Aut}_{\mathbf{EqBr}}(\mathcal{B})$.

This paper is organized as follows. In the remainder of the introduction, we give some indication as to how this work is related to the literature and to physics.

In Section 2 we implement suggestions of Freed [Fre23] to provide a formal definition of a projective TQFT. We define the projectivization functor in Section 2.2. We apply it to various (even higher) Morita categories in Section 2.3, where we also explain some concrete examples of projective targets. The definition of a projective TQFT is in Section 2.4, and the relationship with extensions of bordism categories is discussed in Section 2.5.

In Section 3 we discuss anomalies in general. In Section 3.2 we explicitly describe how a projectivity cocycle classifies a once-categorified invertible TQFT, following [Fre23]. In Sections 3.3 and 3.4 we provide the connection with the sandwich picture in Theorem 3.12 and the "universal anomaly" in Corollary 3.17. Together with Section 2, this proves Theorem A.

In Sections 4.1, 4.2, 4.3, and 4.4 we collect the necessary preliminaries regarding (braided) fusion categories, mostly from [ENO10]. In Section 4.5 we introduce the 3-groups which are processed into anomaly theories, and in Sections 4.6 and 4.7 we explain a detailed analogy between these 3-groups and the groups Pin and Spin.

In Sections 5.1 and 5.3 we review some facts concerning fully-extended 3-dimensional TQFTs and Drinfeld centers, before moving onto anomalies. WRT theories (Theorem D) are discussed in Section 5.2. A projective action of the braided automorphism 2-group of the center (resp. orthogonal group of the underlying metric group) is considered in Section 5.4 (resp. Section 5.5), proving Theorem B (resp. Theorem C), and their avatars as *projective* theories are discussed in Section 5.6. The anomalies/projectivity are translated into the language of topological symmetries in Section 5.8. The functorial assignment of a linear $\mathrm{O}(n, n; \mathbb{F}_p)$-equivariant 3-dimensional TQFT to a vector space over $\mathbb{F}_p$ is considered in Section 5.7.

In Appendix A we recall general facts about TQFTs, mostly from [Lur09]. In particular, we discuss the particular version(s) of the Cobordism Hypothesis we will use in this paper in Section A.1. Relative TQFTs and invertible TQFTs are considered in Sections A.2 and A.3.

Appendix B is dedicated to a discussion of topological symmetry and TQFTs associated to π-finite groupoids. We review some definitions and facts from [FMT22], and define a notion of a "reduction" of topological symmetry in Section B.3.

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1.2. **Anomalies.** Anomalies have been studied for many years, under many different guises. Early appearances include [Ste49, Adl69, BJ69, tH80], and more contemporary resources, which directly inspired this paper, include [Fre23, Fre14a, FHLT10]. See [Fre23] and the references contained

therein for more thorough resources. We take the point of view that an anomaly is a feature, rather than a bug [tH80, Fre23]. Anomalies can, and often are, discussed for more general classes of QFTs, but we will restrict our attention to topological anomalies of TQFTs. In this paper, the anomalies arise from the Postnikov towers of higher automorphism groupoids, as explained in Section 1.1.

This notion of an anomalous theory is closely tied to the notion of a *projective* TQFT, as outlined in [Fre23]. We introduce projective target categories for such theories in Section 2.2. Given any symmetric monoidal (∞, d)-category (not necessarily linear), in Definition 2.3 we give a formal definition for a *projectivization* of any such target. A d-dimensional projective TQFT is then a functor from the bordism category of interest to the projectivization of the target category of interest. In Proposition 3.5 we record the nearly tautological statement that this notion of a projective theory is equivalent to an anomalous theory.

Given any sufficiently nice (see Remark 2.12 for the precise condition) symmetric monoidal (∞, m)-category $\mathcal{S}$, the even higher Morita category [JFS17] $\mathbf{Alg}_n\mathcal{S}$ is a symmetric monoidal $(\infty, m+n)$-category. The projectivization $\mathbb{P}\mathbf{Alg}_n\mathcal{S}$ is a candidate target for projective $(m+n)$-dimensional TQFTs. This gives a well-defined notion of a (fully-extended) projective TQFT in any dimension. An explicit example of interest is the projectivization of **Vect** (considered in Section 2.3.2), which can serve as a target for non-extended TQFTs of any dimension.

Let $\mathcal{T}$ denote the $(d+1)$-dimensional target (i.e. the target for bulk theories). Write $\Omega\mathcal{T} = \mathrm{End}_{\mathcal{T}}(1)$, and Ω^j for the j-fold endomorphisms of the unit. We show in Theorem 2.4 that $\mathbb{P}$ depends naturally on $\mathcal{T}$, and that $\mathbb{P}$ commutes with Ω^j. In particular, if the linear category $\mathcal{T}$ satisfies $\Omega^d\mathcal{T} \simeq \mathbf{Vect}$, and every invertible object of $\Omega^{d-1}\mathcal{T}$ is trivializable (see Warning 1), then we have that:

$$\Omega^{d-1}\mathbb{P}\Omega\mathcal{T} \simeq \mathbb{P}\Omega^d\mathcal{T} \simeq \mathbb{P}\mathbf{Vect} \ .$$

Since $\mathbb{P}\mathbf{Vect}$ is an avatar of the category of projective vector spaces (see Remark 2.14), this is the precise sense of the statement about mapping class group representations in Section 1.1.

Remark 1.1 (Framing anomalies). The results in this paper concerning three-dimensional theories are restricted to the framed setting, in order to avoid the discussion of an SO-fixed point structure on (braided) fusion categories. If a given theory cannot be upgraded from a framed theory to an oriented theory in this manner, then it is said to have a nontrivial *framing anomaly*. (See Example 3.6 for a formal definition in terms of invertible once-categorified theories.) The anomalies in this paper which correspond to projective actions of (higher) groups are not entirely of the same sort as a framing anomaly: The tangential structure on our source $\mathbf{Bord}_d^X$ (borrowing notation from [Lur09]) is the trivial map $X \to B\,\mathrm{O}(d)$.

This is not to say that the discussion in this paper is entirely orthogonal to the discussion of framing anomalies: The four-dimensional Crane-Yetter theory (see Section 5.1) plays an important role in studying the framing anomaly of the 3-dimensional Reshetikhin-Turaev theory [RT90, RT91, Wit89, Wal91, FHLT10].

1.2.1. *Fully-extended anomalous WRT theory.* The anomaly theories of the anomalous 3-dimensional theories are valued in the Morita 4-category of *braided* fusion categories,[5] written **BrFus**. By constructing the projectivization of $\Omega\mathbf{BrFus} \simeq \mathbf{Fus}$ (where **Fus** is the Morita 3-category of fusion categories) we are able to introduce a concrete definition of a *projective* 3*-dimensional TQFT* in Section 5.6.

The objects of $\mathbb{P}\mathbf{Fus}$ are 1-morphisms in **BrFus** from an invertible object to the unit. Spelling this out, an object consists of some invertible object $\mathcal{A}$ of **BrFus**, along with an $\mathbb{E}_1$-algebra object of the monoidal 2-category of $\mathcal{A}$-modules. For instance, the regular module $\mathcal{A}_{\mathcal{A}}$ is such a 1-morphism. As an object of the arrow category $\mathbf{BrTens}^{\downarrow}$ (whose objects are 1-morphisms of **BrTens** and whose

[5] See Section 4.1 for detailed definitions.

morphisms are lax squares [JFS17]), any such regular module defines a fully-dualizable 1-morphism [Haï24].

By the Cobordism Hypothesis, $\mathcal{A}_{\mathcal{A}}$ therefore defines the projective theory $\mathbf{WRT}_{\mathcal{A}}$ of Section 1.1. In words, Theorem D says that we can upgrade WRT and Crane-Yetter to be G-theories in such a way that WRT still lives relative to Crane-Yetter.

Remark 1.2. As in [BJS21, BJSS21], we work with Ind-completions of fusion categories; Remark 4.1 explains why results from the finite abelian setting, e.g. those of [ENO10], still apply.

1.2.2. *An anomaly of a fully-extended framed TV theory.* Let $\mathcal{C}$ be a fusion category and F its framed 3-dimensional TQFT, as in Section 1.1.

As explained in Section 1.1, F is well-defined as a framed theory, and its anomaly appears when we ask it to be equivariant for the symmetries of its Drinfeld center.

The Drinfeld center $\mathcal{Z}(\mathcal{C})$ is an object of the 4-category **BrFus**, which also has duals [BJS21], so this classifies a 4-dimensional TQFT ζ. The Drinfeld center $\mathcal{Z}(\mathcal{C})$ tautologically acts on $\mathcal{C}$, meaning the theory F can be upgraded to a relative theory $F_\zeta\colon \zeta \to 1$. The Drinfeld center turns out to be invertible [BJSS21], and in fact $\mathcal{Z}(\mathcal{C})$ is trivializable in **BrFus**: The module $\mathcal{C}$ in $\mathrm{Hom}_{\mathbf{BrFus}}(\mathcal{Z}(\mathcal{C}), 1)$ is an equivalence. I.e. $F_\zeta\colon \zeta \to 1$ is an equivalence of theories. The anomalous symmetries of the original theory F appear when we consider the symmetries of ζ, or equivalently $\mathcal{Z}(\mathcal{C})$.

In [ENO10], the homotopy type of the 3-type $B\,\mathrm{Aut}_{\mathbf{Fus}}(\mathcal{C})$ is studied. There is a natural map:

$$\mathrm{Aut}_{\mathbf{Fus}}(\mathcal{C}) \to \mathrm{Aut}_{\mathbf{EqBr}}(\mathcal{Z}(\mathcal{C}))\ ,$$

where $\mathrm{Aut}_{\mathbf{EqBr}}$ denotes the 2-group of braided autoequivalences. As it turns out, this map is an equivalence when restricted to the truncation of $\mathrm{Aut}_{\mathbf{Fus}}(\mathcal{C})$ to a 2-group. The top nontrivial homotopy group of $B\,\mathrm{Aut}_{\mathbf{Fus}}(\mathcal{C})$ is $\pi_3 = \mathbf{k}^\times$, so the upshot of this discussion is that the higher automorphism groupoid defines a bundle:

$$\begin{array}{ccccc} B^3\mathbf{k}^\times & \longrightarrow & B\,\mathrm{Aut}_{\mathbf{Fus}}(\mathcal{C}) & \longrightarrow & * \\ \downarrow & & \downarrow & & \downarrow \\ * & \longrightarrow & B\,\mathrm{Aut}_{\mathbf{EqBr}}(\mathcal{Z}(\mathcal{C})) & \xrightarrow{k_3} & B^4\mathbf{k}^\times \end{array} \tag{1.3}$$

The class k_3 classifies the anomaly theory α_{k_3} of Theorem B (Section 3.2), which by construction extends to $\mathbf{Bord}_4^{B\,\mathrm{Aut}_{\mathbf{EqBr}}(\mathcal{Z}(\mathcal{C}))}$, and the original theory F canonically defines a relative theory $F_{k_3}\colon \alpha_{k_3} \to 1$, where 1 denotes the trivial once-categorified theory with source $\mathbf{Bord}_3^{B\,\mathrm{Aut}_{\mathbf{EqBr}}(\mathcal{Z}(\mathcal{C}))}$ (Theorem 5.10). In other words, F is well-defined as an anomalous $\mathrm{Aut}_{\mathbf{EqBr}}(\mathcal{Z}(\mathcal{C}))$-equivariant theory.

If $\mathcal{Z}(\mathcal{C})$ is pointed and $|A|$ is odd, the canonical splitting of $\mathrm{Aut}_{\mathbf{EqBr}}(\mathcal{Z}(\mathcal{C}))$ (Section 4.4.5) lets us (canonically) write its classifying space as a semidirect product of the Eilenberg-Mac Lane spaces $B\pi_1$ and $B^2\pi_2$, where $\pi_1 = \mathrm{O}(A,q)$ and $\pi_2 = A$ (see Section 4.3). Pulling k_3 back along this splitting gives the class $O_4(A,q)$ of Section 1.1, hence an anomaly theory $\alpha_{O_4}\colon \mathbf{Bord}_3^{B\,\mathrm{O}(A,q)} \to \mathbf{BrFus}$ and a relative theory $F_{O_4}\colon \alpha_{O_4} \to 1$.

The homotopy fiber of $O_4(A,q)$ is a $B^3\mathbf{k}^\times$-bundle over $B\,\mathrm{O}(A,q)$. This is the classifying space of a 3-group we call $\mathrm{3Lip}(A,q)$ (Definition 4.46). This is an analogue of the Lipschitz (a.k.a. Clifford) group, reviewed in Section 4.6. As in the classical story, $\mathrm{3Lip}(A,q)$ can be cut down to an extension by a finite group, rather than all scalars. We call this 3Pin (Definition 4.48), and it is analogous to the group Pin. We can restrict to $\mathrm{SO}(A,q) \hookrightarrow \mathrm{O}(A,q)$ to obtain 3Spin (Definition 4.52), an analogue of Spin. This is a part of an analogy in Table 1, which is fleshed out in detail in Section 4.7.

The resulting anomalies are those of Theorem C above (Theorem 5.17 and Corollary 5.21 in the body of this paper).

TABLE 1. A detailed analogy between the Clifford algebra and spin representation associated to (a Lagrangian in a) quadratic vector space, and the braided fusion category and fusion module category associated to (a Lagrangian in a) finite metric group. See Section 4.6 for the former, and Section 4.5 for the latter. On the right we use the split case $A = L \oplus L^*$, $\mathcal{C} = \mathbf{Vect}[L^*]$ (see Proposition 4.29). Here $\mathcal{A}^\times$ denotes the space of invertible objects of $\mathcal{A}$, and $\mu_{l^4} \subset \mathbf{k}^\times$ the group of l^4-th roots of unity, where $l = |L|$. This analogy is explained in detail in Section 4.7.

1-dimensional	3-dimensional
(V, q)	(A, q)
$\mathrm{SO}(V, q) \subset \mathrm{O}(V, q)$	$\mathrm{SO}(A, q) \subset \mathrm{O}(A, q)$
$\mathbf{k}^\times$	$B^2\mathbf{k}^\times$
$\mathrm{Cliff}(V)$	$\mathcal{A} = (\mathbf{Vect}[A], *, \beta_q)$
$\{x, y\} = b_q(x, y)$	$\beta_{b,a} \circ \beta_{a,b} = \langle a, b\rangle_q \,\mathrm{id}$
$V \rtimes \mathrm{O}(V, q)$	$\mathrm{Aut}_{\mathbf{EqBr}}(\mathcal{A})$
$\Gamma = \mathrm{Lip}$	$3\mathrm{Lip}(A, q)$
spinor norm	$N_{(A,q)}$
$\mathrm{Pin}(V, q)$	$3\mathrm{Pin}(A, q)$
$\mathrm{Spin}(V, q)$	$3\mathrm{Spin}(A, q)$
$\{\pm 1\} \hookrightarrow \mathbf{k}^\times$	$B^2\mu_{l^4} \to B^2\mathbf{k}^\times$
See Remark 4.58	$B^2\mathcal{A}^\times$
See Remark 4.58	$\mathrm{Pic}(\mathcal{A})$
$V \simeq L \oplus L^*$	$A \simeq L \oplus L^*$
$\wedge^\bullet L^*$	$\mathcal{C} = (\mathbf{Vect}[L^*], *)$
$\mathrm{End}(\wedge^\bullet L^*) \simeq \mathrm{Cliff}$	$\mathrm{Aut}_{\mathbf{Fus}}(\mathcal{C}) \simeq \mathrm{Pic}(\mathcal{A})$

If one must trivialize the anomaly[6], and it is not trivializable on all of π_1, it can be pulled back along a map $f\colon G \to \pi_1$. This results in an anomaly:

$$\alpha\colon \mathbf{Bord}_3^{BG} \to \mathbf{BrFus}\ ,$$

and F still defines a relative theory $\alpha \to 1$. A *trivialization* is an equivalence: $1 \xrightarrow{\sim} \alpha$, and the *trivialized theory* is the composition:

$$1 \xrightarrow{\sim} \alpha \to 1\ .$$

Note that this is an endomorphism of the trivial once-categorified theory defined on $\mathbf{Bord}_3^{BG}$, and therefore (Proposition A.9) equivalent to a theory

$$F_G\colon \mathbf{Bord}_3^{BG} \to \mathbf{Fus}\ .$$

[6]An anomaly is really "part" of the theory, and only acts as an obstruction when you are "quantizing", see [Fre23, §4].

One should think that we started with a theory F (or fusion category $\mathcal{C}$) and obtained a G-equivariant theory F_G (or fusion category $\mathcal{C}$ along with fully coherent[7] action of G on $\mathcal{C}$ via bimodules).

Remark 1.4. By construction, the direct sum of these bimodules for each $g \in G$ is precisely the G-extension associated to the same data in [ENO10, Theorem 1.3].

Remark 1.5. Sometimes the entire cohomology group containing the obstruction vanishes, meaning the anomaly is trivializable. One example of this is $\mathcal{C} = \mathbf{Vect}[L]$, for L a vector space over $\mathbb{F}_p$. In this case, the (cohomology group and therefore the) obstruction is shown to vanish in [EG18]. In the language of this paper, this means that there is an $\mathrm{O}(n, n; \mathbb{F}_p)$-equivariant theory sending the point to $\mathcal{C}$, where $n = \dim L$ and $\mathrm{O}(n, n; \mathbb{F}_p) = \mathrm{O}(L \oplus L^*, \mathrm{ev})$. This is stated as Corollary 5.29 in the text.

1.2.3. *Characteristic classes.* Anomalies are most interesting when they cannot be trivialized as in Section 5.5.2. This happens, for example when we consider the full symplectic group acting on the quantum harmonic oscillator: The Hilbert space $L^2(\mathbb{R})$ is a nontrivially projective representation of $\mathrm{Sp}_2(\mathbb{R})$, and the projectivity classifies the universal cover $\mathrm{Mp}_2(\mathbb{R}) \to \mathrm{Sp}_2(\mathbb{R})$. The corresponding cocycle classifies (a multiple of) $w_2 \in H^2(B\,\mathrm{Sp}_2(\mathbb{R}))$.

Similarly, finite orthogonal groups have characteristic classes [FP78] and one might wonder if these characteristic classes match the anomaly theories discussed in this paper.

More generally, one might ask which 4-cocycles on a finite group describe anomalies of 3-dimensional TQFTs; Proposition 4.45 and Corollary 5.28 give a partial answer, see Section 1.1.

1.2.4. *Relationship with the literature.* The anomalies studied in this paper are 't Hooft anomalies for discrete internal symmetries of Dijkgraaf-Witten theories. These were first studied (in various dimensions) in [KT14a, KT14b], and (in all dimensions) in the once-extended functorial language in [Mül20, MS20].

In the 3-dimensional and pointed/quasi-trivial case, the obstruction classes in [KT14a, Mül20, MS20] are shown (in [ENO10, §11.8]) to be equivalent to the obstruction classes in [ENO10].

The higher groupoids studied in [ENO10] were studied as symmetries of 3-dimensional theories in [FPSV15, FS16]. The obstruction theory of [ENO10] is used in Section 4.4 to define the 3-groups which characterize the anomaly theories. The same obstruction theory used in this paper from [ENO10] has been used extensively in the literature to study MTCs in the unitary setting [CGPW16, DGP$^+$21, DGP$^+$23].

A partial description of the Brauer-Picard 3-group of the Asaeda-Haagerup fusion categories was given in [GJS15]. The missing information was precisely this k-invariant which classifies the anomaly studied herein, which was shown to vanish for these examples in [GIS18].

The finiteness results of [DSPS20, BJS21, BJSS21, Haï24] are referenced in Section 5.1 to construct fully-extended, and sometimes invertible, TQFTs which agree with the theories associated with certain π-finite spaces (in the sense of [FMT22, §A.2]).

The original Crane-Yetter and Reshetikhin-Turaev invariants/TQFTs [RT90, RT91, TV92, CY93, Del01] are believed to agree with (oriented upgrades) of the theories discussed in Section 5.2 and Section 5.3. In particular, Theorem 5.4 is consistent with the idea that Crane-Yetter theory encodes an anomaly of the Reshetikhin-Turaev theory associated to the MTC [Wal91, Haï25].

The notion of a projective 3-dimensional TQFT is presumably closely related to the notion of a "modular functor" [Seg89, MS89, Tur94, Til98, BK01, BW23], which is roughly a system of projective representations of mapping class groups. Indeed, any projective 3-dimensional TQFT gives rise to a projective mapping class group representation for any surface. A formal comparison would be quite interesting, but we do not pursue this here.

[7] Fully coherent means that the assignment of a bimodule to each group element is not only defined up to isomorphism: it is a functor from the discrete 3-category with objects G to the full 3-group of $\mathcal{C}$-bimodules. See Section 5.5.2 for more.

1.3. **Chern-Simons theory.** Let G be a compact Lie group, and $\lambda \in H^4(BG, \mathbb{Z})$. Consider the Reshetikhin-Turaev TQFT attached to G at level λ [RT90, RT91, BM05, Sti08, FT21]. Recall this sends the circle, with the framing which extends over the disk, to the (semisimplification of the) category of representations of the quantum group at fixed root of unity determined by λ.

When $G = T$ is a torus, write $\Pi = \mathrm{Hom}(\mathrm{U}(1), T)$ for the associated lattice. Then λ determines a nondegenerate symmetric bihomomorphism

$$\langle\, ,\, \rangle : \Pi \times \Pi \to \mathbb{Z} \ .$$

This induces a homomorphism $T \to T^*$ with kernel given by a finite group A. This finite group inherits the quadratic form associated to $\langle\, ,\, \rangle$, and therefore becomes a nondegenerate metric group (A, q) as in Section 4.3.1. See [FHLT10, §9.3] for the role played by this finite group in identifying the framing anomaly of Chern-Simons.

The main upshot of this, for us, is that the braided fusion category $\mathcal{B} = \mathbf{Vect}[A]$ with convolution and braiding from q (see (4.19) in Section 4.3.2) generates the same 1-2-3 theory as the Reshetikhin-Turaev 1-2-3 theory attached to the category of representations of the quantum group for T at level λ [FHLT10].

The problem of extending Chern-Simons theory to the point has received considerable attention [FHLT10, Hen17, FT21]. One way to extend this Reshetikhin-Turaev 1-2-3 theory to the point is by asking for a fusion category $\mathcal{C}$ such that $\mathcal{B} \cong \mathcal{Z}(\mathcal{C})$ as braided categories. Given that one such fusion category $\mathcal{C}$ exists, one might also wonder how uniquely it is determined. It is shown, in [ENO11], that two fusion categories are Morita equivalent if and only if the Drinfeld centers are equivalent as braided categories. This is strengthened in [ENO10] to an equivalence of 2-groups:

$$\tau_{\leq 1} \mathrm{Aut}_{\mathbf{Fus}}(\mathcal{C}) \simeq \mathrm{Aut}_{\mathbf{EqBr}}(\mathcal{Z}(\mathcal{C})) \ .$$

I.e. the automorphisms of $\mathcal{C}$ in **Fus** are almost determined by the braided autoequivalences of $\mathcal{Z}(\mathcal{C})$; as explained after Theorem B, the lack of coherence at the top level is captured by the class k_3 (Section 4.4.3) and by the anomaly of Theorem B.

1.4. **Anomalies in the Langlands program.** An analogous anomaly to the one studied here appears in the Langlands program [AV23, BZSV24]. Indeed, this was the main source of motivation for the project. More specifically, upgrading *Rozansky-Witten theory* [RW97, RW10, KRS09, KR10] to a boundary/relative theory for certain 4-dimensional gauge theories would produce a version of the relevant relative theory on the B-side (i.e. spectral side). The author hopes to return to these questions in the future.

1.5. **Gapped systems.** It is well-known that gapped phases of matter are described at low energy/long range by (unitary) TQFTs [Fre14b, FH21, FT21]. See also [RW18, KZ22]. In particular, it is pointed out in [Fre23, Interlude (P.12)], that the low energy linear theory describing the system is in fact *not* topological (it has a metric dependence [Wit89]), however its projectivization is indeed topological. We discuss projective 3-dimensional TQFTs in Section 5.6.

't Hooft anomalies for discrete internal symmetries are particularly important for the study of SPT phases [Kap14, KT14a, KT14b, Mül20, MS20].

A related subject is the study of anyon systems. One way to model anyon systems, spelled out e.g. in [RW18], is using unitary modular categories (UMC). The question of whether or not the UMC is a Drinfeld center, which plays a big role in this paper, is also important in the study of anyon systems. For example, [RW18, Conjecture 4.2] states that a UMC being realized as a topological phase of matter is equivalent to being a Drinfeld center of a (unitary) fusion category.

When the fusion category $\mathcal{C}$ is the category of vector spaces graded by a finite group L, possibly twisted by a cocycle, the corresponding 3-dimensional theory is Dijkgraaf-Witten theory [DW90]. Kitaev introduced a Hamiltonian realization [Kit03], and for $L = \mathbb{Z}/2$ this is the famous toric code.

The structure of the 3-type $B\,\mathrm{Aut}_{\mathbf{Fus}}(\mathcal{C}) \simeq B\,\mathrm{Pic}(\mathcal{Z}(\mathcal{C}))$, as identified in [ENO10], is the source of the projectivity/anomalies identified in Section 5. The homotopy type of this groupoid is of

much interest in the literature. For example, let $\mathcal{B}$ be an arbitrary unitary modular category. It is conjectured in [AWH22] that the homotopy type of $B\operatorname{Pic}(\mathcal{B})$ is related to a space of gapped Hamiltonians which give rise to the topological order described by $\mathcal{B}$.

1.6. **Sandwiches and anomalies.** The discussion in Section 1.2.2 fits into a more general picture, which relates to the language of topological symmetries [FMT22]. An upshot of this discussion, in the context of Section 1.2.2, will be that the anomalous theory in Theorem 5.17 (resp. Corollary 5.21) will equivalently define a $\sigma_{B\,\mathrm{O}(A,q)}$-module structure (in the sense of [FMT22], see Appendix B) twisted by the class which classifies the 3-group 3Pin (resp. 3Spin) over $\mathrm{O}(A,q)$.

More generally, let $\mathcal{T}$ be an $(\infty, d+1)$-category with duals, let $F\colon \mathbf{Bord}_d \to \Omega\mathcal{T}$ be a TQFT, and suppose that the classifying space $\widetilde{X}$ of the higher automorphism group of $F(*)$ is a $B^d\mathbf{k}^\times$-bundle over a π-finite space X, classified by $c\colon X \to B^{d+1}\mathbf{k}^\times$, as in (1.3). Then c defines an anomaly theory $\alpha_c\colon \mathbf{Bord}_d^X \to \mathcal{T}$ (Section 3.2), and integrating over maps to X [FMT22, §A.2] defines a $(d+1)$-dimensional theory $\sigma^{d+1}_{X,c}$. The question of whether F descends from $\mathbf{Bord}_d^{\widetilde{X}}$ to $\mathbf{Bord}_d^X$ can then be phrased in several equivalent ways, summarized in Theorem 3.12 (whose part (iii) is Theorem A(iii)).

Example 1.6. We have discussed this formalism in general, but it is most interesting when the space X has a novel interpretation. E.g. in Sections 5.4 and 5.5 we use the canonical identification of the truncation to a 2-type with the braided autoequivalences of the center [ENO10] to interpret this as an anomaly of F as a theory which is equivariant with respect to the braided autoequivalences of the center.

Slightly more generally, consider any nondegenerate braided fusion category $\mathcal{A}$. There is a fibration:

$$\begin{array}{ccc} B^3\mathbf{k}^\times & \longrightarrow & B\operatorname{Pic}(\mathcal{A}) \\ \downarrow & & \downarrow{\scriptstyle \tau_{\leq 2}} \\ * & \longrightarrow & B\operatorname{Aut}_{\mathbf{EqBr}}(\mathcal{A}) \end{array}$$

(see Definition 4.2) by [ENO10, Theorem 5.2], which is classified by some map $c\colon B\operatorname{Aut}_{\mathbf{EqBr}}(\mathcal{A}) \to B^4\mathbf{k}^\times$.

For any monoidal functor $\Phi\colon G \to \operatorname{Aut}_{\mathbf{EqBr}}(\mathcal{A})$, one expects $\mathbf{WRT}_{\mathcal{A}}$ to obtain a $\left(\sigma^4_{BG,\Phi^* c}, \rho_{BG,\Phi^* c}\right)$-module structure (Remark B.16). The question of trivializing the anomaly is the question of $\sigma^4_{BG,\Phi^* c}$ having a Neumann boundary theory (see Section 1.7), which is equivalent to whether or not c pulls back to define the trivial cohomology class on BG.

So there are many different four-dimensional theories in the story: the untwisted pure topological G-gauge theory (which we *want* to be defined relative to, in order to gauge), and then the Crane-Yetter theory and the twisted topological G-gauge theory (which we are *always* on the boundary of).

One way to think of the universal such theory $\sigma^4_{B\operatorname{Aut}_{\mathbf{EqBr}}(\mathcal{A}),c}$ is as the result of gauging the canonical anomaly-free G-action on $\mathbf{CY}_{\mathcal{A}}$ defined by Φ.

1.7. **Gauging and anomalies.** Given a quantum field theory F and a (compact Lie) group G, under certain conditions, a new theory, the *G-gauged theory*, can be produced via a procedure known as *gauging*. This has received various mathematical formulations, e.g. for topological theories see [Tel14, §2.3]. Also see [FMT22, §3.4]. For non-extended versions of some of the 3-dimensional theories considered in this paper, see [Mül20, MS20]. This procedure cannot always be carried out, and the obstructions are known as *'t Hooft anomalies* [tH80].

Gauging is often phrased as a two step process:

(1) couple F to a background G-field (principal G-bundle),
(2) integrate over all G-bundles to obtain a new ordinary theory: the gauged theory.

In the context of topological field theories, the first step is to extend[8] F from $\mathbf{Bord}_d^{\mathrm{fr}}$ (or e.g. $\mathbf{Bord}_d^{\mathrm{or}}$) to $\mathbf{Bord}_d^{BG}$.

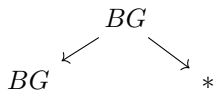
Remark 1.7. Even before proceeding to the second step, one can see how anomalies of the sort defined in Section 3.2 can act as obstructions to gauging. One should imagine that we cannot quite compatibly extend F to $\mathbf{Bord}_d^{BG}$: We can only extend it as a relative theory. I.e. we have a (once-categorified) theory α (which *is* defined on $\mathbf{Bord}_d^{BG}$) and a relative theory $F\colon \alpha \to 1$. This becomes a TQFT defined on $\mathbf{Bord}_d^{BG}$ only once we provide a trivialization $1 \xrightarrow{\sim} \alpha$.

As in Section 1.6, a theory defined on $\mathbf{Bord}_d^{BG}$ defines a boundary theory $\widetilde{F}\colon 1 \to \sigma_{BG}^{d+1}$. (see Section B.1 for notation, and Section 3.2 for a more detailed discussion).

The second step is easier to formulate in this language. Namely, as in [FMT22, Definition 3.22, Example 3.24], the process of integrating over all G-bundles is pairing with the *Neumann boundary theory*:

$$F/_\epsilon \sigma \coloneqq \epsilon \otimes_{\sigma_{BG}^{d+1}} \widetilde{F} \ , \tag{1.8}$$

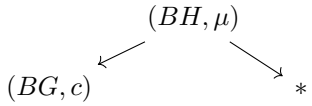
where the Neumann boundary theory ϵ is the morphism induced (as in Proposition B.12) by the augmentation:



Note that if the theory σ_{BG}^{d+1} is twisted by a cocycle c, then we will not necessarily have an augmentation map: We need to trivialize c. I.e. if we have a boundary theory

$$\widetilde{F}\colon 1 \to \sigma_{BG,c}^{d+1} \ ,$$

in order to gauge the G-action, we need to pair with an augmentation for BG. One obtains an augmentation from any correspondence of the form



where $H \subset G$ is a subgroup, and μ is a trivialization of $c|_H$. This is, for example, the form of the classification of simple, fully extended 2-dimensional topological theories relative to gauge theory in [FT22]: subgroups equipped with central extensions.

As is explained in Section 1.6, the cocycle c classifies an anomaly theory α_c. So a trivialization of α_c (i.e. of c) determines an augmentation, which induces the Neumann boundary condition, which allows us to gauge as in (1.8). In other words: α_c obstructs gauging.

2. Projective TQFTs

2.1. **Preliminaries: arrow categories.** Let $\mathcal{T}$ be a symmetric monoidal (∞, n)-category. We use the lax arrow category $\mathcal{T}^{\downarrow}$ of [JFS17, Definition 5.14], which is again a symmetric monoidal (∞, n)-category [JFS17, Corollary 6.9]. Following [JFS17, §1], we think of the objects of $\mathcal{T}^{\downarrow}$ as

[8] There is always a trivial extension of F to $\mathbf{Bord}_d^{BG}$. The pullback of any extension along $\mathbf{Bord}_d \to \mathbf{Bord}_d^{BG}$ (sending every bordism to the same bordism equipped with the product principal G-bundle) is an ordinary theory with internal G-symmetry. Usually one has a given (internal) action of G on F, and one insists that the extension agrees with this action after pulling back.

"vertical" arrows, i.e. 1-morphisms $f\colon x \to y$ of $\mathcal{T}$, and the 1-morphisms $f \to f'$ of $\mathcal{T}^{\downarrow}$ are squares

$$\begin{array}{ccc} x & \xrightarrow{a} & x' \\ {\scriptstyle f}\downarrow & \swarrow & \downarrow{\scriptstyle f'} \\ y & \xrightarrow[b]{} & y' \end{array}$$

where the 2-morphism $b \circ f \Rightarrow f' \circ a$ is not necessarily invertible. The higher morphisms are squares of squares, and so on, as in [JFS17, Definition 5.14]. There are source and target functors $s, t\colon \mathcal{T}^{\downarrow} \to \mathcal{T}$, sending $f\colon x \to y$ to x and y respectively. A lax natural transformation $\eta\colon F \Rightarrow G$ between functors $F, G\colon \mathcal{B} \to \mathcal{T}$ is a functor $\eta\colon \mathcal{B} \to \mathcal{T}^{\downarrow}$ such that $s \circ \eta = F$ and $t \circ \eta = G$ [JFS17, Definition 1.3]. We write $\mathcal{T}[0] \to \mathcal{T}$ for the functor picking out the unit, where $\mathcal{T}[0] \simeq *$ [JFS17, Definition 6.10].

An anomalous theory $\alpha \to 1$ is, by definition (see Remark A.11), a functor

$$\overline{Z}\colon \mathbf{Bord}_d \to \mathcal{T}^{\downarrow} \tag{2.1}$$

such that

$$s \circ \overline{Z} = \alpha \qquad\qquad t \circ \overline{Z} = 1\ .$$

Remark 2.2. As explained in Remark A.11, the relative theories in this paper are lax natural transformations. Everything can be repeated to produce an oplax version by replacing $\mathcal{T}^{\downarrow}$ with $\mathcal{T}^{\rightarrow}$ everywhere.

We write $\mathcal{T}^{\downarrow 1}$ for the arrow category 'terminating at the unit' (compare [JFS17], where arrows originating at the unit are used):

$$\mathcal{T}^{\downarrow 1} \coloneqq \mathcal{T}^{\downarrow} \overset{h}{\underset{\mathcal{T}}{\times}} \mathcal{T}[0] \qquad\qquad \begin{array}{ccc} \mathcal{T}^{\downarrow 1} & \longrightarrow & \mathcal{T}[0] \\ \downarrow & & \downarrow \\ \mathcal{T}^{\downarrow} & \xrightarrow{t} & \mathcal{T} \end{array}\ .$$

Similarly, we will define the subcategory 'originating at an invertible object' to be:

$$\mathcal{T}^{\times\downarrow} \coloneqq \mathcal{T}^{\times} \overset{h}{\underset{\mathcal{T}}{\times}} \mathcal{T}^{\downarrow} \qquad\qquad \begin{array}{ccc} \mathcal{T}^{\times\downarrow} & \longrightarrow & \mathcal{T}^{\downarrow} \\ \downarrow & & \downarrow{\scriptstyle s} \\ \mathcal{T}^{\times} & \longrightarrow & \mathcal{T} \end{array}\ .$$

2.2. **The projectivization of a target category.** The projectivization of the category $\Omega\mathcal{T}$, written $\mathbb{P}\Omega\mathcal{T}$, consists of arrows in $\mathcal{T}$ between an invertible object of $\mathcal{T}$ and the unit in $\mathcal{T}$. For instance, for $\mathcal{T} = \mathbf{Alg}$ an object is a module over an Azumaya algebra, which up to Morita equivalence is a vector space defined only up to scalar; see Section 2.3.2.

Definition 2.3. Define the *projectivization* of a symmetric monoidal (∞, n)-category (thought of as an $\mathbb{E}_\infty$-algebra among complete n-fold Segal spaces) $\mathcal{T}$ to be:

$$\mathbb{P}\Omega\mathcal{T} \coloneqq \mathcal{T}[0] \overset{h}{\underset{\mathcal{T}}{\times}} \mathcal{T}^{\downarrow} \overset{h}{\underset{\mathcal{T}}{\times}} \mathcal{T}^{\times}$$

Here the first fiber product is taken along the unit $\mathcal{T}[0] \to \mathcal{T}$ and the target functor $t\colon \mathcal{T}^{\downarrow} \to \mathcal{T}$, and the second along the source functor $s\colon \mathcal{T}^{\downarrow} \to \mathcal{T}$ and the inclusion $\mathcal{T}^{\times} \to \mathcal{T}$. I.e. $\mathbb{P}\Omega\mathcal{T}$ consists of arrows from an invertible object to the unit, as in (2.1). Recall from [JFS17, Def. 6.10] that $\mathcal{T}[0] \simeq *$.

Warning 1. This construction does *not* only depend on $\Omega\mathcal{T}$, as the notation might suggest. It depends on the group of isomorphism classes of invertible objects of $\mathcal{T}$. For example, many categories are of the form $\Omega\mathcal{T}_1 \simeq \Omega\mathcal{T}_2$ where $\pi_0\mathcal{T}_1^{\times}$ is trivial and $\pi_0\mathcal{T}_2^{\times}$ is not. We nonetheless keep the notation $\mathbb{P}\Omega\mathcal{T}$, which should always be read as a function of $\mathcal{T}$.

This becomes especially important when we study 3-dimensional projective targets in Section 2.3.3. See Remark 2.16.

Theorem 2.4. *There is a well-defined functor from the ∞-groupoid $\mathbb{E}_\infty(n\text{-}\mathbf{cat})$ of symmetric monoidal (∞,n)-categories (and symmetric monoidal equivalences) to itself,*

$$\mathbb{P}\colon \mathbb{E}_\infty(n\text{-}\mathbf{cat}) \to \mathbb{E}_\infty(n\text{-}\mathbf{cat})$$

which sends a symmetric monoidal (∞,n)-category $\mathcal{T}$ to $\mathbb{P}\Omega\mathcal{T}$, thought of as $\mathbb{E}_\infty$-algebras among complete n-fold Segal spaces.

Furthermore, projectivization commutes with iterated looping, i.e. the following diagram commutes for all integers $j \in \mathbb{Z}$ satisfying $1 \le j \le n$:

$$\begin{array}{ccc} \mathbb{E}_\infty(n\text{-}\mathbf{cat}) & \xrightarrow{\mathbb{P}} & \mathbb{E}_\infty(n\text{-}\mathbf{cat}) \\ \downarrow{\scriptstyle \Omega^j} & & \downarrow{\scriptstyle \Omega^j} \\ \mathbb{E}_\infty((n-j)\text{-}\mathbf{cat}) & \xrightarrow{\mathbb{P}} & \mathbb{E}_\infty((n-j)\text{-}\mathbf{cat}) \end{array} \tag{2.5}$$

Proof. The functor $(-)^{\downarrow}$ is defined in [JFS17, Definition 5.14], and shown to depend naturally on both arguments in [JFS17, Corollary 5.19], which is extended to the symmetric monoidal case in [JFS17, Corollary 6.9]. Rephrasing this slightly, this says that $(-)^{\downarrow}$ defines a functor from the ∞-groupoid of symmetric monoidal (∞,n)-categories and equivalences to itself.

There is a commuting diagram of three homotopy fiber squares:

$$\begin{array}{ccccc} \mathbb{P}\Omega\mathcal{T} & \longrightarrow & \mathcal{T}^{\downarrow 1} & \xrightarrow{t} & \mathcal{T}[0] \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{T}^{\times\downarrow} & \longrightarrow & \mathcal{T}^{\downarrow} & \xrightarrow{t} & \mathcal{T} \\ \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle s} & & \\ \mathcal{T}^{\times} & \longrightarrow & \mathcal{T} & & \end{array} \tag{2.6}$$

because

$$\left(\mathcal{T}^{\downarrow 1}\right) \underset{\mathcal{T}^{\downarrow}}{\overset{h}{\times}} \left(\mathcal{T}^{\times\downarrow}\right) \simeq \mathcal{T}[0] \underset{\mathcal{T}}{\overset{h}{\times}} \mathcal{T}^{\downarrow} \underset{\mathcal{T}}{\overset{h}{\times}} \mathcal{T}^{\times} \ .$$

Therefore the functoriality of $\mathbb{P}$ follows from the functoriality of $(-)^{\downarrow}$ [JFS17] and the functoriality of $(-)^{\times}$.

For a fixed non-negative integer n, we will first show the statement for $j = 1$:

$$\Omega\mathbb{P}\Omega(-) \simeq \mathbb{P}\Omega^2(-) \ . \tag{2.7}$$

Recall that, by definition (here $\mathbb{P}\Omega$ is applied to $\mathcal{T}$, and to $\Omega\mathcal{T}$ respectively):

$$\Omega\mathbb{P}\Omega\mathcal{T} = \Omega\left(\mathcal{T}[0] \underset{\mathcal{T}}{\overset{h}{\times}} \mathcal{T}^{\downarrow} \underset{\mathcal{T}}{\overset{h}{\times}} \mathcal{T}^{\times}\right)$$

and

$$\mathbb{P}\Omega^2\mathcal{T} = \Omega\mathcal{T}[0] \underset{\Omega\mathcal{T}}{\overset{h}{\times}} (\Omega\mathcal{T})^{\downarrow} \underset{\Omega\mathcal{T}}{\overset{h}{\times}} (\Omega\mathcal{T})^{\times} \ .$$

There is a canonical equivalence of symmetric monoidal complete n-fold Segal spaces ([JFS17, Proposition 6.12]):

$$\Omega\mathcal{T} \simeq \mathcal{T}[0] \underset{\mathcal{T}}{\overset{h}{\times}} \mathcal{T}^{\downarrow} \underset{\mathcal{T}}{\overset{h}{\times}} \mathcal{T}[0] \ ,$$

which means that (2.7) follows from the fact that homotopy limits commute with each other, together with the facts that Ω commutes with $(-)^{\downarrow}$ and with $(-)^{\times}$.

To see the statement for general $j \in \{1, \dots, n\}$, apply (2.7) to $\Omega^{j-1}\mathcal{T}$. □

Remark 2.8. The $\mathbb{P}$ in Definition 2.3 can be thought of as standing for "projective", as $\mathbb{P}\mathbf{Vect}$ should be thought of as some version of the category of projective spaces. Therefore it can also be thought of as standing for "pure states", as these form the projectivization of the Hilbert space of mixed states.

As is conjectured by Freed in [Fre23], the projectivization should have a description as a directed pullback of symmetric monoidal (∞, n)-categories

$$\begin{array}{ccc} \mathbb{P}\Omega\mathcal{T} & \overset{F}{\dashrightarrow} & \mathcal{T}^{\times} \\ \vdots & & \downarrow \\ * & \longrightarrow & \mathcal{T} \end{array} \tag{2.9}$$

One way to interpret (2.9), is as expressing "exactness" at a term of an "exact sequence" of symmetric monoidal ∞-categories. We do not attempt to make this notion precise here, however consider the following diagram of functors:

$$\begin{array}{ccccccc} \cdots & \Omega\Omega\mathcal{T}^{\times} & \longrightarrow & \Omega\Omega\mathcal{T} & \longrightarrow & \mathbb{P}\Omega\Omega\mathcal{T} & \\ & \hookrightarrow \Omega\mathcal{T}^{\times} & \longrightarrow & \Omega\mathcal{T} & \longrightarrow & \mathbb{P}\Omega\mathcal{T} & \cdots \end{array} \tag{2.10}$$

where the connecting homomorphism is F from (2.9), with $\Omega\mathcal{T}$ in place of $\mathcal{T}$. The functors $\Omega\Omega\mathcal{T} \to \mathbb{P}\Omega\Omega\mathcal{T}$ and $\Omega\mathcal{T} \to \mathbb{P}\Omega\mathcal{T}$ in (2.10) are the canonically defined functors which send an object to the same object regarded as an endomorphism of the unit. We hope to return to this in future work.

Remark 2.11. Note that, if every invertible object of $\mathcal{T}$ is trivializable, then the functor $\Omega\mathcal{T} \to \mathbb{P}\Omega\mathcal{T}$ is essentially surjective. Otherwise, there are objects in $\mathbb{P}\Omega\mathcal{T}$ outside of the image of $\Omega\mathcal{T}$: for example, if $\mathcal{T}$ consists of algebra objects in $\Omega\mathcal{T}$, consider any nontrivial invertible object of $\mathcal{T}$. Then the regular module defines an object of $\mathbb{P}\Omega\mathcal{T}$ which is not in the image of $\Omega\mathcal{T}$.

2.3. **Fully-extended projective target categories.**

2.3.1. *Projective Morita categories.* As discussed in Warning 1, the construction of $\mathbb{P}\Omega\mathcal{T}$ actually depends on $\mathcal{T}$, rather than merely $\Omega\mathcal{T}$. However, if we are given a symmetric monoidal ($\otimes$-sifted-cocomplete) (∞, n)-category $\mathcal{S}$, we can form a projectivization of $\mathcal{S}$ as follows (see Remark 2.12 for a discussion of different models of the higher Morita category). We can form the symmetric monoidal $(\infty, n+1)$-category $\mathcal{T} = \mathbf{Alg}(\mathcal{S})$, the "even higher" Morita category of $\mathbb{E}_1$-algebras [JFS17].

Recall that $\Omega\mathcal{T} \simeq \mathcal{S}$, so in this case we obtain a motto about projective objects: *An object of* $\mathbb{P}(\mathcal{S}) \coloneqq \mathbb{P}\Omega\mathbf{Alg}(\mathcal{S})$ *can be thought of as an invertible (e.g. Azumaya) algebra in* $\mathcal{S}$ *along with a module over this algebra (also internal to* $\mathcal{S}$*).*

More generally, we obtain a projective target category by taking the projectivization of the Morita $(\infty, n+m)$-category $\mathbf{Alg}_n(\mathcal{S})$ for a (sufficiently nice) symmetric monoidal (∞, m)-category $\mathcal{S}$.

Remark 2.12 (Models for Morita categories). The higher Morita (∞, n)-category of $\mathbb{E}_n$-algebras in a $\otimes$-sifted-cocomplete (resp. $\otimes$-GR-cocomplete) symmetric monoidal (∞, m)-category $\mathcal{S}$ was constructed in [Sch14] (resp. [Hau17]). This was extended to the "even higher" Morita $(\infty, n+m)$-category, written $\mathbf{Alg}_n\mathcal{S}$, in [JFS17].

There are expected relationships between the models in [Sch14] and [Hau17], but the author is unaware of a theorem relating them. One benefit of the former is that n-dualizability is known [GS18, Theorem 5.1][9], but the arguments do not translate to the model in [Hau17].

On the other hand, the results used in Section 5.1 concerning the dualizability and invertibility of (braided) fusion categories are from [BJS21, BJSS21], which use the model from [Hau17].

2.3.2. *The projectivization of the category of vector spaces.* Let $\mathcal{S} = \mathbf{Vect}$ be the symmetric monoidal category of finite-dimensional vector spaces over $\mathbf{k}$. Setting $\mathcal{T} = \mathbf{Alg}$ to be the symmetric monoidal Morita 2-category of associative algebras, we can calculate $\mathbb{P}\mathbf{Vect}$ to be the 2-category given by "all modules over all Azumaya algebras". The 1-morphisms are (op)lax squares

$$\begin{array}{ccc} 1 & = & 1 \\ {\scriptstyle {}_A M}\downarrow & \overset{\varphi}{\nearrow} & \downarrow{\scriptstyle {}_B N} \\ A & \xrightarrow{\sim} & B \end{array}$$

where A and B are Morita-invertible algebras with modules M and N. The data of the filling φ is a linear map between M and N which intertwines the module structures according to an invertible Morita morphism $A \xrightarrow{\sim} B$. The following are examples of objects of $\mathbb{P}\mathbf{Vect}$.

- Any fully-dualizable (i.e. separable) algebra as a module over itself.
- If there is an algebra map to the trivial algebra (an augmentation map) then this defines a module structure on the one-dimensional vector space.
- The rank n vector space as a module over the matrix algebra M_n.

Remark 2.13. Note that $\mathbb{P}\mathbf{Vect}$ serves as a target for non-extended projective TQFTs in any dimension. The category $\mathbb{P}\mathbf{Vect}$ is used in [Haï25] to formalize the notion of anomalous non-extended non-semisimple Witten-Reshetikhin-Turaev theory.

Remark 2.14. There is a related 2-category, written **Proj**, which is defined in [Fre23, Appendix A]. The objects are finite-dimensional vector spaces, the 1-morphisms are linear maps, and there is a 2-morphism between any two linear maps $\varphi, \psi \colon V \to W$ for every $\lambda \in \mathbf{k}^\times$ such that $\varphi(v) = \lambda\psi(v)$ for all $v \in V$.

There is an equivalence $\mathbf{Proj} \to \mathbb{P}\mathbf{Vect}$, e.g. as symmetric monoidal bicategories ([Haï25, Remark 7.2]), which sends a vector space V to itself regarded as an endomorphism of the unit in **Alg**, i.e. as a $\mathbf{k}$-bimodule.

2.3.3. *The projectivization of the Morita category of monoidal categories.* Let $\mathcal{T} = \mathbf{BrFus}$ be the Morita 4-category of braided fusion categories (see Section 4.1) and recall:

$$\mathbf{Fus} \simeq \Omega\mathcal{T} = \mathrm{End}_{\mathcal{T}}(1) \ .$$

Applying $\mathbb{P}$ from Definition 2.3 to **BrFus**, we obtain the following.

Corollary 2.15. $\mathbb{P}\mathbf{Fus}$ *is a symmetric monoidal* 4*-category. The objects of* $\mathbb{P}\mathbf{Fus}$ *are arrows in* **BrFus** *which are between an invertible object and the unit (i.e. central modules over invertible braided fusion categories). The* 1*-morphisms in* $\mathbb{P}\mathbf{Fus}$ *are (op)lax squares:*

$$\begin{array}{ccc} 1 & = & 1 \\ {\scriptstyle {}_{\mathcal{A}}\mathcal{M}}\downarrow & \overset{\Phi}{\nearrow} & \downarrow{\scriptstyle {}_{\mathcal{B}}\mathcal{N}} \\ \mathcal{A} & \xrightarrow{\sim} & \mathcal{B} \end{array}$$

where $\mathcal{A}$ *and* $\mathcal{B}$ *are invertible objects of* **BrFus**, $\mathcal{M}$ *and* $\mathcal{N}$ *are (possibly non-invertible)* 1*-morphisms in* **BrFus**, *and the filling* Φ *is a* 2*-morphism in* **BrFus**, *i.e. a 'twisted' intertwining bimodule between* $\mathcal{M}$ *and* $\mathcal{N}$. *The higher morphisms are defined by general principles in* [JFS17].

[9] Note however that the only $(n+1)$-dualizable object is the unit [GS18, Theorem 6.1].

Remark 2.16. Recall from Warning 1 that the construction of $\mathbb{P}\Omega\mathcal{T}$ actually depends on the group of isomorphism classes of invertible objects of $\mathcal{T}$, rather than just $\Omega\mathcal{T}$. In low dimensions this didn't appear, unless we were considering super-vector spaces, because the category of finite-dimensional vector spaces, the Morita category of algebras, and the Morita category of fusion categories all have trivial group of isomorphism classes of invertible objects, i.e. all invertible objects are trivializable.

This is not true in this dimension: The Morita 4-category **BrFus** has a highly nontrivial group $\pi_0\mathbf{BrFus}^\times$, which is known to be equivalent to the so-called Witt group of braided fusion categories [BJSS21, Theorem 4.2]. Note that this contains the Witt group of $\mathbb{Q}$, and is therefore certainly nontrivial. This clearly produces a different projectivization in comparison with performing the same construction, say, in the arrow category of the 4-category $B\mathbf{Fus}$, which would clearly not detect any of the non-trivializable invertible objects of **BrFus**.

2.4. **Projective theories.**

Definition 2.17. A d-dimensional *projective* (X,ζ)*-TQFT* is a symmetric monoidal functor:

$$\overline{F}\colon \mathbf{Bord}_d^{(X,\zeta)} \to \mathbb{P}\Omega\mathcal{T}\ .$$

Write $s\colon \mathbb{P}\Omega\mathcal{T} \to \mathcal{T}^\times$ for the composition of the vertical functors on the left in (2.6). Given such a theory $\overline{F}$, composing with the functor s results in an invertible theory:

$$\alpha\colon \mathbf{Bord}_d^{(X,\zeta)} \xrightarrow{\overline{F}} \mathbb{P}\Omega\mathcal{T} \xrightarrow{s} \mathcal{T}^\times\ .$$

This theory is the *projectivity* of the projective theory $\overline{F}$. Sometimes to emphasize that a theory is *not* projective, we will call it *linear*.

Proposition 2.18. *A trivialization of the projectivity $\alpha = s\circ\overline{F}$ determines a linear theory $F_X\colon \mathbf{Bord}_d^X \to \Omega\mathcal{T}$. We will call F_X a* linearization *of $\overline{F}$.*

Proof. A projective TQFT is equivalently a natural transformation from an invertible TQFT to the trivial TQFT. Composing this with the trivialization results in an endomorphism of the trivial theory valued in $\mathcal{T}$ which, by Proposition A.9, is equivalent to a TQFT valued in $\Omega\mathcal{T}$. □

Definition 2.19. Assume a projective TQFT $\overline{F}$ has a linearization upon restriction to framed bordisms, i.e. there is a theory F^{fr} such that the following diagram commutes:

(2.20)
$$\begin{array}{ccccc}
\mathbf{Bord}_d^{\mathrm{fr}} & \hookrightarrow & \mathbf{Bord}_d^X & & \\
\downarrow{\scriptstyle F^{\mathrm{fr}}} & \searrow^{\overline{F}^{\mathrm{fr}}} & \downarrow{\scriptstyle \overline{F}} & \searrow^{\alpha} & \\
\Omega\mathcal{T} & \longrightarrow & \mathbb{P}\Omega\mathcal{T} & \longrightarrow & \mathcal{T}^\times
\end{array}$$

Then we call F^{fr} the *underlying framed theory of* $\overline{F}$.

2.5. **Extensions of the bordism category.** Just as in classical representation theory, a projective representation is equivalent to a linear representation of an extension. The notion of an extension of a bordism category by a modular functor already appeared in Segal's original paper [Seg04, Definition 5.2].

The projectivity of a projective TQFT in the sense of Definition 2.17 is also captured by an extension of a bordism category as follows. Namely, the extension is the universal object over which the projectivity can be trivialized. In slightly more detail, it is the pullback:

(2.21)
$$\begin{array}{ccccc}
\widetilde{\mathbf{Bord}_d^X} & \longrightarrow & \mathbf{Bord}_d^X & & \\
\downarrow{\scriptstyle \widetilde{F}} & & \downarrow{\scriptstyle \overline{F}} & \searrow^{\alpha} & \\
\Omega\mathcal{T} & \longrightarrow & \mathbb{P}\Omega\mathcal{T} & \longrightarrow & \mathcal{T}^\times
\end{array}$$

and since the composition $\Omega\mathcal{T} \to \mathbb{P}\Omega\mathcal{T} \to \mathcal{T}^\times$ is canonically trivialized, so is the projectivity of $\widetilde{F}$, i.e. the extension is the universal object over which the projectivity is trivializable.

This description as a pullback of symmetric monoidal ∞-categories is not completely precise. A more formal description is as follows: Given a projective TQFT $\overline{F}\colon \mathbf{Bord}_d^X \to \mathbb{P}\Omega\mathcal{T}$, the projectivity $\alpha = s \circ \overline{F}$ is an invertible TQFT, and therefore classifies a map of groupoids $\widetilde{\alpha}$ (A.14) as explained in Section A.3. The homotopy fiber $\mathbf{E}$ of $\widetilde{\alpha}$ is some extension of the (groupoid given by inverting all morphisms in the) original bordism category:

$$\begin{array}{ccccc} \Omega\mathcal{T}^\times & \longrightarrow & \mathbf{E} & \longrightarrow & * \\ \downarrow & & \downarrow & & \downarrow \\ * & \longrightarrow & \left|\mathbf{Bord}_d^X\right| & \xrightarrow{\alpha} & \mathcal{T}^\times \end{array}$$

and the claim is that $\mathbf{E}$ is the result of inverting all morphisms in some extension of the original bordism category.

Remark 2.22. Recall from Remark 2.11 that the map $\Omega\mathcal{T} \to \mathbb{P}\Omega\mathcal{T}$ might not be essentially surjective, and therefore the map from $\widetilde{\mathbf{Bord}}_d^X$ to $\mathbf{Bord}_d^X$ might not be essentially surjective either.

One instance of this appearing in examples is the filled bordism category, where bordisms (of any codimension) are equipped with a filling, i.e. a bordism of one higher dimension of which they are the boundary (so in particular these fillings are manifolds with corners); see [Haï24, Definition 4.1] and [Haï25, Definition 6.1], where this category is attributed to Walker [Wal06]. For example, the point is not an object of the fully-extended filled bordism category, but S^0 is.

The bordism category $\mathbf{Bord}_d^{\widetilde{X}}$ provides a description for (a fully-extended version of) the extension of the source discussed in [Fre23, §3]. See also Remark 3.10.

3. Anomalies

See [Fre23] and the references therein for more details on anomalies.

3.1. **Definitions.** Fix the target $\mathcal{T}$ from Section A.1. Let (X, ζ) be as in Section A.1.1, and recall we have defined (X, ζ)-theories as functors out of $\mathbf{Bord}_d^{(X,\zeta)}$.

Definition 3.1. An *anomaly* α is an invertible, once-categorified TQFT:

$$\alpha\colon \mathbf{Bord}_d^{(X,\zeta)} \to \mathcal{T} \ .$$

A *d-dimensional anomalous TQFT F with anomaly α* is a TQFT defined relative to α:

$$F_\alpha\colon \alpha \to 1 \ ,$$

where 1 denotes the trivial theory on $\mathbf{Bord}_d^{X,\zeta}$.

A *trivialization of the anomaly* α is an equivalence $1 \xrightarrow{\sim} \alpha$. The *trivialized theory* is the composition:

$$1 \xrightarrow{\sim} \alpha \xrightarrow{F} 1$$

which defines a (X, ζ)-TQFT of dimension d by Proposition A.9.

Remark 3.2. Sometimes anomalies are defined to be boundary theories rather than relative theories (see Remark A.13).

Sometimes anomalies are defined as left boundary/relative theories $1 \to \alpha$, rather than right boundary/relative theories: $\alpha \to 1$. In this case trivializations would be $\alpha \xrightarrow{\sim} 1$.

Given an (X, ζ)-theory Z, write Z_{fr} for the underlying framed theory given by restricting Z to bordisms with trivial (X, ζ)-structure.

Definition 3.3. We say a framed TQFT:

$$F\colon \mathbf{Bord}_d^{\mathrm{fr}} \to \Omega\mathcal{T}$$

has an anomaly as an (X,ζ)*-theory* if there is an anomaly theory

$$\alpha\colon \mathbf{Bord}_d^{(X,\zeta)} \to \mathcal{T}$$

and a trivialization $t\colon 1_{\mathrm{fr}} \xrightarrow{\sim} \alpha_{\mathrm{fr}}$, along with an anomalous theory

$$F_\alpha\colon \alpha \to 1_{(X,\zeta)}$$

(where $1_{(X,\zeta)}$ denotes the trivial (X,ζ)-theory) and an equivalence:

$$\theta\colon F \simeq (F_\alpha)_{\mathrm{fr}} \circ t\ ,$$

where the endomorphism $(F_\alpha)_{\mathrm{fr}} \circ t$ of 1_{fr} is regarded as a theory valued in $\Omega\mathcal{T}$ by Lemma A.8 (as in Proposition A.9).

Definition 3.4. An (X,ζ)*-anomaly of the theory* F is such a quadruple $(\alpha, F_\alpha, t, \theta)$. The trivialization t and equivalence θ are often canonically defined, in which case we will write that $\alpha \xrightarrow{F_\alpha} 1_X$ is the (X,ζ)-anomaly of F.

Proposition 3.5. *A projective theory with projectivity* α *(in the sense of Definition 2.17) is naturally equivalent to an anomalous TQFT with anomaly* α *(in the sense of Definition 3.1).*

Proof. An anomalous TQFT is a natural transformation from an invertible TQFT to the unit. This is, by definition, a functor from the bordism category to the arrow category such that composition with the source and target functors give you an invertible object and the unit respectively. This is precisely the same condition as factoring through $\mathbb{P}\Omega\mathcal{T} \to \mathcal{T}^{\downarrow}$, i.e. defining a projective TQFT. □

Example 3.6 (Framing anomaly). Let $(X,\zeta) = (B\,\mathrm{SO}\,(d)\,, \zeta_\iota)$ be as in Example A.5, i.e. $\mathbf{Bord}_d^{(X,\zeta)} \simeq \mathbf{Bord}_d^{\mathrm{or}}$. A framed theory F is often said to have a nontrivial *framing anomaly*, if it cannot be upgraded to an oriented theory, i.e. a $(B\,\mathrm{SO}\,(d)\,, \zeta_\iota)$-theory. This is equivalent to having a $(B\,\mathrm{SO}\,(d)\,, \zeta_\iota)$-anomaly as in Definition 3.4 such that the associated anomaly theory α is a nontrivial theory.

Example 3.7. Let $(X,\zeta) = (BG, \zeta_{\mathrm{triv}})$ be as in Example A.7. A BG-anomaly as in Definition 3.4 (or G-anomaly) for the theory F is an anomalous action of G on F. We will see in Theorem 3.12 that this is equivalent to F having a $\left(\sigma^{d+1}_{BG,c}, \rho\right)$-module structure (Section B.2) where the twist c is closely related to the anomaly theory.

3.2. **The anomaly associated to a projectivity class.** Let X be a space (higher groupoid) equipped with a cocycle c classifying $\widetilde{X}$:

$$\begin{array}{c} \widetilde{X} \\ \downarrow{\scriptstyle \pi} \\ X \xrightarrow{c} B^{d+1}\mathbf{k}^\times \end{array}$$

Let $\Omega\mathcal{T} = \mathrm{End}_{\mathcal{T}}(1)$ denote the looping of the fixed target $\mathcal{T}$ from the beginning of Section A.1. The anomaly theory α_c is defined as follows. By the Cobordism Hypothesis (A.6), an X-theory $\mathbf{Bord}_{d+1}^X \to \mathcal{T}$ is determined by a functor from X to $\mathcal{T}^\sim$, so we can define α_c to be the restriction to $\mathbf{Bord}_d^X$ of the theory classified by the following composition of functors:

$$X \xrightarrow{c} B^{d+1}\mathbf{k}^\times \hookrightarrow \mathcal{T}\ .$$

By definition, α_c factors through "scalars" in the target, which is sufficient to be an invertible TQFT. I.e. a relative theory $\alpha_c \to 1_X$ is an anomalous X-theory with anomaly α_c.

Proposition 3.8. *An anomalous X-theory $F_\alpha\colon \alpha_c \to 1_X$ determines an $\widetilde{X}$-theory $\widetilde{F}\colon \mathbf{Bord}_d^{\widetilde{X}} \to \Omega\mathcal{T}$.*

Proof. The relative theory F_α is a once-categorified d-dimensional theory valued in the arrow category $\mathcal{T}^{\downarrow}$. By the Cobordism Hypothesis (A.6) for once-categorified theories (see the paragraph after (A.6)) applied to $\mathcal{T}^{\downarrow}$ (see [JFS17, Theorem 7.15]), F_α is classified by a functor from X to $\left(\mathcal{T}^{\downarrow}\right)^{\sim}$ with source α_c and target 1, i.e. by a $B^d\mathbf{k}^\times$-equivariant map of bundles over X (as in Hypothesis Q, where $B^d\mathbf{k}^\times$ acts on $\Omega\mathcal{T}^\sim$ by scalars):

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{|F_\alpha|} & X \times \Omega\mathcal{T}^\sim \\ & \searrow^{\pi} & \swarrow_{p_1} \\ & X & \end{array}$$

Composing with the projection to the second factor we obtain a map

$$\widetilde{X} \xrightarrow{|F_\alpha|} X \times \Omega\mathcal{T}^\sim \xrightarrow{p_2} \Omega\mathcal{T}^\sim \ .$$

The result follows from the Cobordism Hypothesis (A.6): Every TQFT $\widetilde{F}\colon \mathbf{Bord}_d^{\widetilde{X}} \to \Omega\mathcal{T}$ is classified by such a map $\widetilde{X} \to \Omega\mathcal{T}^\sim$. □

Example 3.9. Fix $d = 1$ and let c be a cocycle classifying a central extension $\widetilde{G}$ of a group G by $\mathbf{k}^\times$. If $X = BG$ and $\widetilde{X} = B\widetilde{G}$, then a projective representation of G with projectivity cocycle c (i.e. a linear representation of $\widetilde{G}$) defines such a theory $\widetilde{F}$.

Remark 3.10. Recall that the upshot of the discussion in Section 2.5 is that every projective TQFT is an honest (i.e. non-anomalous) functor out of an extension of the original bordism category. According to Proposition 3.5, the same is true for anomalous theories. This is the familiar statement that equipping your bordisms with extra structure allows one to trivialize the anomaly. See Remark 5.6 for more on this extended category in the context of Reshetikhin-Turaev and Crane-Yetter TQFTs.

The bordism category $\mathbf{Bord}_d^{\widetilde{X}}$ provides a description for (a fully-extended version of) the extended bordism category for 't Hooft anomalies described by such a cocycle c.

Note that, as discussed in Remark 2.22, the functor from the extended bordism category to the original one is not necessarily essentially surjective, e.g. it might consist of "filled" bordisms if the anomalous theory in question does not extend to the point. For instance, in this case the point is not an object of the extended category, but S^0 is.

3.2.1. *Trivializing the anomaly.* We have a symmetric monoidal functor $\pi_*\colon \mathbf{Bord}_d^{\widetilde{X}} \to \mathbf{Bord}_d^X$ given by composing the map defining the $\widetilde{X}$-structure on a bordism with the map π.

Question 1. Does $\widetilde{F}$ factor as follows?

$$\begin{array}{ccc} \mathbf{Bord}_d^{\widetilde{X}} & \xrightarrow{\widetilde{F}} & \Omega\mathcal{T} \\ \downarrow{\scriptstyle \pi_*} & \nearrow_{F_X} & \\ \mathbf{Bord}_d^X & & \end{array}$$

Question 1 is equivalent to the question of splitting $\widetilde{X} \to X$ (i.e. trivializing c). We can ask a weaker question, by pulling back along a map $f\colon Y \to X$. Write $f^*\widetilde{X}$ for the pullback:

$$\begin{array}{ccc} f^*\widetilde{X} & \longrightarrow & \widetilde{X} \\ \downarrow{\scriptstyle f^*\pi} & & \downarrow{\scriptstyle \pi} \\ Y & \xrightarrow{f} & X \end{array}$$

Write $f^*\widetilde{F}$ for the corresponding $f^*\widetilde{X}$-theory:

$$f^*\widetilde{F}\colon \mathbf{Bord}_d^{f^*\widetilde{X}} \to \mathbf{Bord}_d^{\widetilde{X}} \xrightarrow{\widetilde{F}} \Omega\mathcal{T} \ .$$

Now we can state a refinement of Question 1.

Question 2. Does $f^*\widetilde{F}$ factor as follows?

$$\begin{array}{ccc} \mathbf{Bord}_d^{f^*\widetilde{X}} & \xrightarrow{f^*\widetilde{F}} & \Omega\mathcal{T} \\ {\scriptstyle (f^*\pi)_*}\downarrow & \nearrow_{F_Y} & \\ \mathbf{Bord}_d^{Y} & & \end{array}$$

Question 2 is similarly equivalent to splitting $f^*\widetilde{X} \to Y$, or equivalently trivializing f^*c.

A trivialization $1 \xrightarrow{\sim} \alpha_c$ would provide us with an X-theory by Proposition A.9. However if c is not trivializable, the best we will be able to do is to trivialize the pullback along some map $f\colon Y \to X$. This allows us to regard the anomaly theory α_c as a Y-theory:

$$\begin{array}{ccc} \mathbf{Bord}_d^{Y} & & \\ \downarrow{\scriptstyle f} & \searrow^{\alpha_{f^*c}} & \\ \mathbf{Bord}_d^{X} & \xrightarrow{\alpha_c} & \mathcal{T} \end{array} \tag{3.11}$$

A trivialization of $\widetilde{X}$ over Y (i.e. of f^*c) uniquely determines a trivialization $1_Y \xrightarrow{\sim} \alpha_c^Y$. Composing with the anomalous theory gives $1_Y \xrightarrow{\sim} \alpha_c^Y \to 1_Y$, which is a Y-theory by Proposition A.9.

3.3. **Anomalies and sandwiches.** Let $\mathcal{T}$ be the $(d+1)$-dimensional target fixed in Section A.1. The d-dimensional TQFTs we will consider will be valued in the looping $\Omega\mathcal{T}$. Let X be a (pointed, connected) π-finite space. Consider a cocycle $c\colon X \to B^{d+1}\mathbf{k}^\times$ classifying $\widetilde{X}$:

$$\begin{array}{ccc} \widetilde{X} & & \\ \downarrow{\scriptstyle \pi} & & \\ X & \xrightarrow{c} & B^{d+1}\mathbf{k}^\times \ . \end{array}$$

Let α_c denote the anomaly theory from Section 3.2, and let $\sigma_{X,c}$ denote the theory from Section B.1.

Theorem 3.12. *An X-anomaly $F_c\colon \alpha_c \to 1_X$ of F (as in Definition 3.4) naturally defines:*

- *(i) A projective X-theory $\overline{F}\colon \mathbf{Bord}_d^X \to \mathbb{P}\Omega\mathcal{T}$ with projectivity α_c and underlying theory F (as in Definitions 2.17 and 2.19).*
- *(ii) An $\widetilde{X}$-theory $\widetilde{F}\colon \mathbf{Bord}_d^{\widetilde{X}} \to \Omega\mathcal{T}$ whose underlying framed theory is F.*
- *(iii) Assuming Hypothesis Q, a $(\sigma_{X,c}, \rho_{X,c})$-module structure on F (Section B.2).*

Proof. The fact that F_c determines (i) follows from Proposition 3.5. Similarly, Proposition 3.8 shows that F_c determines (ii). Finally, the fact that F_c determines (iii) follows from Proposition B.14 (which depends on Hypothesis Q). □

Theorem 3.13. *Consider the setup of Theorem 3.12, and assume that the cocycle c is trivializable after being pulled back along a map $f\colon Y \to X$. A trivialization of f^*c naturally defines a trivialization of the anomaly theory α_{f^*c} (as in (3.11)) which in turn defines:*

- *(i) A linearization of $\overline{F}$ (as in Proposition 2.18).*
- *(ii) A Y-theory F_Y which factors $f^*\widetilde{F}$ (as in Question 2).*

(iii) Assuming Hypothesis Q, a reduction (Definition B.24) from the $(\sigma_{X,c}, \rho_{X,c})$-module structure in Theorem 3.12 (iii) to a (σ_Y, ρ_Y)-module structure.

Furthermore, the absolute Y-theories obtained from (i), (ii), and (iii) agree with the one obtained from the trivialization of the anomaly.

Note that if one such trivialization exists, then the collection of trivializations forms a torsor over $H^d(Y, \mathbf{k}^\times)$.

Proof. In terms of Question 2, a trivialization of $\widetilde{X}$ over Y (i.e. a trivialization of f^*c) explicitly defines a section $\mathbf{Bord}_d^Y \to \mathbf{Bord}_d^{f^*\widetilde{X}}$, which we can compose with $f^*\widetilde{F}$ to obtain the desired theory F_Y.

The theory α_{f^*c} (from (3.11)) is defined by the map f^*c, so a trivialization of f^*c automatically provides a trivialization $1_Y \xrightarrow{\sim} \alpha_{f^*c}$. This produces an automorphism of the trivial Y-theory $1_Y \xrightarrow{\sim} \alpha_{f^*c} \to 1_Y$, which is a theory on $\mathbf{Bord}_d^Y$ by Proposition A.9. This Y-theory is classified (via the Cobordism Hypothesis (A.6)) by the functor $Y \to \Omega\mathcal{T}$ given by the composition of the section of $f^*\widetilde{X}$ with the functor $\widetilde{X} \to \Omega\mathcal{T}^\sim$ classifying the anomalous theory:

$$Y \to f^*\widetilde{X} \to \Omega\mathcal{T}^\sim \ .$$

This agrees with the functor $Y \to \Omega\mathcal{T}^\sim$ classifying F_Y by Theorem 3.12. The agreement between the trivialization of the anomaly and the linearization (i) is Proposition 2.18.

A trivialization of f^*c determines a reduction to $\left(\sigma_Y^{d+1}, \rho_Y\right)$ by Proposition B.23. In particular, the boundary theory $1 \to \sigma_{\widetilde{X}}^{d+1}$ becomes reduced to a boundary theory $1 \to \sigma_Y^{d+1}$ which defines a Y-theory by Proposition B.14. This agrees with F_Y by the commuting square (B.7) in Remark B.5 associated to $X_1 = X$, $X_2 = \widetilde{X}$, and the morphism in (B.22). □

3.4. **A universal anomaly.** Let

$$F\colon \mathbf{Bord}_d^{\mathrm{fr}} \to \Omega\mathcal{T}$$

be a TQFT sending the point to an object $\mathcal{S}$ of $\Omega\mathcal{T}$. Let $\widetilde{X} = B\operatorname{Aut}_{\Omega\mathcal{T}}(\mathcal{S})$. Because we are assuming $\Omega^{d+1}\mathcal{T} \simeq \mathbf{k}$ (see the beginning of Section A.1), we know that $\mathbf{k}^\times$ maps to $\Omega^{d-1}\operatorname{Aut}(\mathcal{S})$, however it is only an equivalence when $\mathcal{S}$ satisfies a sort of simplicity/irreducibility.

Definition 3.14. We say an object $\mathcal{S}$ of $\Omega\mathcal{T}$ is a *Schur object* if $\Omega^{d-1}\operatorname{Aut}(\mathcal{S}) \cong \mathbf{k}^\times$.

If $\mathcal{S}$ is a Schur object, then the top level of the Postnikov tower provides a projectivity class c on the truncation to a $(d-1)$-type:

$$\begin{array}{c} B\operatorname{Aut}_{\Omega\mathcal{T}}(\mathcal{S}) \\ \downarrow{\scriptstyle \pi} \\ \pi_{\le d-1} B\operatorname{Aut}_{\Omega\mathcal{T}}(\mathcal{S}) \xrightarrow{c} B^{d+1}\mathbf{k}^\times \ . \end{array} \tag{3.15}$$

The following follows from Theorems 3.12 and 3.13 by setting

$$\widetilde{X} = B\operatorname{Aut}_{\Omega\mathcal{T}}(\mathcal{S}) \ , \qquad X = \pi_{\le d-1} B\operatorname{Aut}_{\Omega\mathcal{T}}(\mathcal{S}) \ , \text{ and} \qquad Y = BG \tag{3.16}$$

for some (possible higher) group G.

Corollary 3.17. *Let $\mathcal{T}$ be an $(\infty, d+1)$-category with duals which satisfies $\Omega^{d+1}\mathcal{T} \simeq \mathbf{k}$, and let $\mathcal{S}$ be a Schur object (Definition 3.14) of $\Omega\mathcal{T}$.*

- *The class c from (3.15) uniquely determines the following.*
  - *(1) The gerbe $B\operatorname{Aut}_{\Omega\mathcal{T}}(\mathcal{S}) \to X$ of objects of $\Omega\mathcal{T}$ equivalent to $\mathcal{S}$.*
  - *(2) The anomaly theory α_c (as in Section 3.2), which is the projectivity (as in Definition 2.17) of the projective theory in (iii) below.*
  - *(3) The quiche $(\sigma_{X,c}, \rho_{X,c})$ (as in Section B.2).*
- *The object $\mathcal{S}$ uniquely determines the following.*

(i) *An $\widetilde{X}$-theory* $\widetilde{F}\colon \mathbf{Bord}_d^{B\operatorname{Aut}(\mathcal{S})} \to \Omega\mathcal{T}$.
(ii) *An X-anomaly* $F_c\colon \alpha_c \to 1$ *(as in Definition 3.4).*
(iii) *A projective theory* $\overline{F}$ *with projectivity* α_c *(as in Definition 2.17).*
(iv) *Assuming Hypothesis Q, a* $(\sigma_{X,c}, \rho_{X,c})$*-module structure on* F *(as in Section B.2).*

- *Given* $f\colon BG \to X$ *for some (possibly higher) group* G*, a trivialization of* f^*c *determines the following.*
  (a) *A theory* F_{BG} *which factors* $\widetilde{F}$ *(as in Question 2).*
  (b) *A trivialization of the anomaly* α_{f^*c} *(as in (3.11)).*
  (c) *A linearization of the projective theory* $\overline{F}$ *(as in Proposition 2.18).*
  (d) *Assuming Hypothesis Q, a reduction to a* (σ_{BG}, ρ_{BG})*-module structure on* F *(as in Definition B.24).*

  Furthermore, the trivialization in (b), the linearization in (c), and the reduction in (d) determine three BG-theories, which all agree with F_{BG}. If such a trivialization exists, then the trivializations of f^*c form a torsor over $H^d(BG, \mathbf{k}^\times)$.

4. Fusion categories

4.1. **Preliminaries.** Fix an algebraically closed field $\mathbf{k}$ of characteristic zero. Let $\mathbf{Pr}$ be the symmetric monoidal 2-category of presentable $\mathbf{k}$-linear categories with colimit preserving functors (and natural transformations).[10]

As in [BJS21, §3], we will consider the "even higher" Morita category [JFS17] of $\mathbb{E}_2$-categories. This Morita category of braided monoidal categories was proposed in [Wal06, §9], [DSPS20], and [BZBJ18]. See Remark 2.12 for more details on the specific model of the Morita category used here.

The Morita 3-category of tensor categories, $\mathbf{Tens} = \mathbf{Alg}_1(\mathbf{Pr})$, consists of the following:

- the objects are tensor categories,
- the 1-morphisms are bimodule categories,
- the 2-morphisms are functors between bimodule categories, and
- the 3-morphisms are natural transformations.

We will focus on the subcategory $\mathbf{Fus}$ consisting of fusion categories, semisimple bimodule categories, compact-preserving cocontinuous bimodule functors, and natural transformations. It was shown in [BJS21] that this forms a subcategory.

Following [BJS21], we define "fusion" as follows. Given an $\mathbb{E}_1$-algebra object of $\mathbf{Pr}$, there is an increasingly strict list of finiteness conditions one can insist upon:

- cp-rigid: every compact projective object is dualizable.
- compact-rigid: every compact object is dualizable.
- finite: compact rigid and the underlying category is finite.
- fusion: finite, semisimple,[11] with simple unit.

Remark 4.1. Recall the original notions of rigid, finite, and fusion abelian categories as in [EO04, ENO05, EGNO15]. As explained in [BJS21, BJSS21] this setting is related to the one defined above by an Ind-completion, in the following sense. The ambient category of abelian categories is not suitable for TQFT constructions, because it is not closed under the (Deligne) tensor product. Instead, as in [Shi20], we can work in the ambient category of finitely co-complete $\mathbf{k}$-linear categories with right exact functors and natural equivalences, written $\mathbf{Rex}$.

As is explained in [BZBJ18, §3], taking the Ind-completion defines an equivalence between $\mathbf{Rex}$ and the subcategory $\mathbf{Pr}_c \subset \mathbf{Pr}$ consisting of compactly generated $\mathbf{k}$-linear presentable categories,

[10] A category is presentable (sometimes called locally presentable) if it is accessible (generated under colimits by a small subcategory) and cocomplete (closed under small colimits).

[11] By semisimple, we mean [BJS21, Definition 2.27]: Every object is a (possibly infinite) direct sum of simple objects.

compact functors, and natural equivalences.[12] As in [BZBJ18, §3], Ind and the functor taking compact objects of a presentable category define an equivalence of $(2,1)$-categories $\mathbf{Rex} \simeq \mathbf{Pr}_c$.

As it turns out, if an abelian category is rigid, then its Ind-completion is compact-rigid in the above sense [BJS21, Remark 1.6]. Similarly, if an abelian category is finite (in the usual sense [EO04]) then its Ind-completion is finite (compact-rigid and finite underlying category). Indeed, a finite abelian category is equivalent to the category R-mod of finite-dimensional modules over a finite-dimensional algebra R [EGNO15, Definitions 1.8.5 and 1.8.6], so its Ind-completion is the category R-Mod of all R-modules, whose compact objects are the finite-dimensional modules.

The upshot of this discussion is that the Ind-completion of a fusion category in the traditional sense lands in the category $\mathbf{Fus} \subset \mathbf{Alg}_1(\mathbf{Pr})$ as it is defined above. Note that the Deligne-Kelly tensor product coincides with the Deligne tensor product of finite abelian categories (and commutes with taking ind-completions), see the proof of [BJS21, Definition-Proposition 4.8]. Therefore Ind identifies the Morita 3-category of fusion categories in the abelian sense with $\mathbf{Fus} \subset \mathbf{Alg}_1(\mathbf{Pr})$, and any statement about the former which is invariant under equivalence holds for the latter. In particular, we will use various facts from [DSPS20, ENO10] (which are in the abelian setting) throughout.

4.1.1. *Braided fusion categories.* Similarly, braided tensor categories form a 4-category $\mathbf{BrTens} = \mathbf{Alg}_2(\mathbf{Pr})$:

- the objects are braided tensor categories,
- the 1-morphisms are associative algebra objects in the category of bimodule categories,
- the 2-morphisms are bimodule categories between bimodule categories,
- the 3-morphisms are functors between bimodule categories, and
- the 4-morphisms are natural transformations.

The subcategory **BrFus** consists of braided fusion categories, fusion categories equipped with central structures, finite semisimple bimodule categories, compact-preserving cocontinuous bimodule functors, and bimodule natural transformations. It was shown in [BJS21] that this forms a subcategory.

Braided fusion categories also form a 2-category:

- the objects are braided fusion categories,
- the 1-morphisms are functors preserving the braided structure, and
- the 2-morphisms are natural transformations.

The 2-groupoid given by the invertible part of this 2-category is written **EqBr**, as in [ENO10].

4.2. **Higher groupoids attached to a braided fusion category.** Let $\mathcal{A}$ be a braided fusion category. The braided equivalences of $\mathcal{A}$ form a 2-group $\mathrm{Aut}_{\mathbf{EqBr}}(\mathcal{A})$. From any 2-group, we can construct its classifying space $B\,\mathrm{Aut}_{\mathbf{EqBr}}(\mathcal{A})$, which is a 2-type. I.e. it has two nontrivial homotopy groups.

To $\mathcal{A}$ we can also attach the 2-category of $\mathcal{A}$-module categories. The braiding defines an embedding from the 2-category of $\mathcal{A}$-modules to the 2-category of $\mathcal{A}$-bimodules. Therefore the category of $\mathcal{A}$-modules inherits a monoidal structure from the natural one on the category of $\mathcal{A}$-bimodules.

Definition 4.2. The invertible $\mathcal{A}$-modules, written $\mathrm{Pic}(\mathcal{A})$, comprise the *Picard 3-group of* $\mathcal{A}$.

From any 3-group, we can construct its classifying space $B\,\mathrm{Pic}(\mathcal{A})$, which is a 3-type. I.e. it has three nontrivial homotopy groups.

Remark 4.3. Some authors write the full 3-group/type as $B\underline{\underline{\mathrm{Pic}}}(\mathcal{A})$, and write the truncation to a 2-type as $B\underline{\mathrm{Pic}}(\mathcal{A})$ and to a 1-type as $B\,\mathrm{Pic}(\mathcal{A})$.

Instead, we write $B\,\mathrm{Pic}(\mathcal{A})$ for the full 3-group, and write $\tau_{\leq 2}B\,\mathrm{Pic}(\mathcal{A})$ and $\tau_{\leq 1}B\,\mathrm{Pic}(\mathcal{A})$ for the truncations.

[12] A functor is compact if it preserves compact objects.

Theorem ([ENO10, Theorem 5.2])**.** *For a nondegenerate braided fusion category $\mathcal{A}$ we have an equivalence:*

$$\pi_{\leq 2} B\operatorname{Pic}(\mathcal{A}) \simeq B\operatorname{Aut}_{\mathbf{EqBr}}(\mathcal{A}) \ . \tag{4.4}$$

The homotopy groups of $B\operatorname{Pic}(\mathcal{A})$, e.g. from [ENO10, Proposition 7.5] are as follows: π_1 is given by the ordinary group of isomorphism classes of braided equivalences, and π_2 is given by the group of tensor isomorphisms of the identity functor on $\mathcal{A}$. The top homotopy group, which is not involved in (4.4), is $\pi_3 B\operatorname{Pic}(\mathcal{A}) = \mathbf{k}^\times$.

4.2.1. *Braided fusion categories which are Drinfeld centers.* Let $\mathcal{C}$ be a fusion category. To this we can attach the monoidal 2-category $\operatorname{End}_{\mathbf{Fus}}(\mathcal{C})$.

Definition 4.5. The *Brauer-Picard 3-group* of $\mathcal{C}$ is $\operatorname{Aut}_{\mathbf{Fus}}(\mathcal{C})$.

Recall the Drinfeld center of a monoidal category $\mathcal{C}$ is a braided monoidal category, written $\mathcal{Z}(\mathcal{C})$.

Theorem ([EO04])**.** *There is an equivalence of 2-categories:*

$$\operatorname{End}_{\mathbf{Fus}}(\mathcal{C}) \xrightarrow{\sim} \mathcal{Z}(\mathcal{C})\text{-}\mathbf{mod} \ .$$

Passing to the invertible part, we obtain:

$$\operatorname{Aut}_{\mathbf{Fus}}(\mathcal{C}) \simeq \operatorname{Pic}(\mathcal{Z}(\mathcal{C})) \ . \tag{4.6}$$

Remark 4.7. In [ENO11], it is shown that two fusion categories are Morita equivalent if and only if their Drinfeld centers are braided equivalent. This was strengthened in [ENO10]: they prove that there is a fully-faithful embedding of groupoids $\mathbf{EqBr} \to \mathbf{Fus}^\times$. Indeed, combining (4.4) and (4.6) we obtain

$$\pi_{\leq 2} B\operatorname{Aut}_{\mathbf{Fus}}(\mathcal{C}) \simeq B\operatorname{Aut}_{\mathbf{EqBr}}(\mathcal{Z}(\mathcal{C})) \ . \tag{4.8}$$

So the center determines the Morita class of the fusion category itself. However, since the equivalence (4.4) is with the truncated part of $\operatorname{Aut}_{\mathbf{Fus}}(\mathcal{C})$, there is a subtle lack of "coherence", captured by an obstruction class originally studied in [ENO10] and surveyed in Section 4.4.3.

Related results were shown in [KZ18] and [KK12, §5].

In summary: given a braided monoidal category $\mathcal{A}$, we have a 3-type $B\operatorname{Pic}(\mathcal{A})$ attached to it. If it happens to be the case that $\mathcal{A} = \mathcal{Z}(\mathcal{C})$ for some fusion category $\mathcal{C}$, then this 3-type is the classifying space of the Brauer-Picard 3-group of $\mathcal{C}$. All together we have:

$$\begin{array}{ccc}
\operatorname{End}_{\mathbf{Fus}}(\mathcal{C}) & \xrightarrow[\text{[EO04]}]{\sim} & \mathcal{Z}(\mathcal{C})\text{-}\mathbf{mod} \\
\\
B\operatorname{Aut}_{\mathbf{Fus}}(\mathcal{C}) & \xrightarrow{\sim} & B\operatorname{Pic}(\mathcal{Z}(\mathcal{C})) \\
\downarrow & & \downarrow \\
\pi_{\leq 2} B\operatorname{Aut}_{\mathbf{Fus}}(\mathcal{C}) & \xrightarrow{\sim} & \pi_{\leq 2} B\operatorname{Pic}(\mathcal{Z}(\mathcal{C})) \\
& \searrow^{\sim} \quad \swarrow_{\sim}^{\text{[ENO10]}} & \\
& B\operatorname{Aut}_{\mathbf{EqBr}}(\mathcal{Z}(\mathcal{C})) &
\end{array}$$

4.3. **Pointed braided fusion categories.** A fusion category is *pointed* if all simple objects are invertible.

Example 4.9. Let L be a finite group. Consider the category of L-graded vector spaces, $\mathbf{Vect}[L]$, with convolution, i.e. for $a, b, c \in L$:

$$(W * W')_a = \bigoplus_{a=bc} W_b \otimes W_c \ .$$

This category has simple objects $\mathbf{k}_a$ for $a \in L$, and is therefore pointed. Given a 3-cocycle τ on L, we can define a variant of this category $\mathbf{Vect}[L]^\tau$, in which the associators are multiplied by τ [EGNO15, Example 2.3.8]. This category is still pointed.

All pointed fusion categories are of the form $\mathbf{Vect}[L]^\tau$ [EGNO15, Proposition 4.10.3]. (Note however that the twisted version may have no fiber functor. This is a higher analogue of the fact that not all algebras have an augmentation map.)

4.3.1. *Metric groups.* Metric groups [DGNO10, ENO10] play an important role in the theory of pointed braided fusion categories. Let $\mathbf{k}$ be an algebraically closed field of characteristic zero, and let A be a finite abelian group. Write $A^* \coloneqq \mathrm{Hom}(A, \mathbf{k}^\times)$ for the character dual of A.

Definition 4.10. A biadditive map

$$\langle \cdot, \cdot \rangle : A \times A \to \mathbf{k}^\times$$

is a *symmetric bicharacter* if for all $a, b \in A$ we have $\langle a, b \rangle = \langle b, a \rangle$. By acting on the first argument, such a bicharacter $\langle \cdot, \cdot \rangle$ defines a homomorphism $A \to A^*$. If this is an isomorphism, then we say that $\langle \cdot, \cdot \rangle$ is *nondegenerate*.

Definition 4.11. A function

$$q \colon A \to \mathbf{k}^\times$$

is a *quadratic form* on A if

(a) for all $a \in A$ we have $q(a) = q(-a)$, and
(b) the following expression defines a symmetric bicharacter:

$$\langle a, b \rangle_q \coloneqq \frac{q(a+b)}{q(a)\, q(b)} \ .$$

We say q is *nondegenerate* if the bicharacter $\langle \cdot, \cdot \rangle_q$ is nondegenerate.

Remark 4.12. If $|A|$ is odd, then sending q to $\langle \cdot, \cdot \rangle_q$ defines a bijection between quadratic forms and bicharacters.

Definition 4.13. Any pair (A, q) is a *pre-metric group*. A pair (A, q) is called a *metric group* if q is nondegenerate.

Definition 4.14. The *orthogonal group* of a metric group (A, q) is:

$$\mathrm{O}(A, q) \coloneqq \{ f \in \mathrm{Hom}(A, A) \mid q \circ f = q \} \ .$$

Definition 4.15. Let (A, q) be a metric group. Define the *determinant*

$$\det \colon \mathrm{O}(A, q) \to \mathbb{Q}^\times{}_{>0} / \left(\mathbb{Q}^{\times\, 2}_{>0} \right)$$

by sending $g \in \mathrm{O}(A, q)$ to the image of $|(g-1)A| \in \mathbb{N}$.

The *special orthogonal group* is

$$\mathrm{SO}(A, q) \coloneqq \ker(\det) \ . \tag{4.16}$$

4.3.2. *Classification of pointed braided fusion categories.* It is shown in [JS93] that pointed braided fusion categories are classified by finite abelian groups A (the group of simple objects) equipped with a quadratic form $q\colon A \to \mathbf{k}^\times$ which is not necessarily nondegenerate. See also [EGNO15, §8.4].

Example 4.17. If L is any finite abelian group, then the finite abelian group $L \oplus L^*$ has a non-degenerate quadratic form given by evaluation, so $(L \oplus L^*, \mathrm{ev})$ is a metric group. The symmetric bicharacter corresponding to evaluation is:

$$\langle (\ell, \chi), (\ell', \chi') \rangle = \chi(\ell')\,\chi'(\ell) \ .$$

Given an abelian group A, it is shown in [EML54, Theorem 26.1] that $H^4\left(B^2 A, \mathbf{k}^\times\right)$ is isomorphic to the group of quadratic forms $A \to \mathbf{k}^\times$.

Eilenberg and Mac Lane defined an explicit chain complex which computes this cohomology [ML52, EML50a, EML50b, EML51]. They call this abelian group cohomology, and write it as H^*_{ab}:

$$H^n_{\mathrm{ab}}(A, B) = H^{n+1}(K(A, 2), B)$$

for any two abelian groups A and B. I.e. we have

$$H^3_{\mathrm{ab}}\left(A, \mathbf{k}^\times\right) = H^4\left(B^2 A, \mathbf{k}^\times\right) \cong \mathrm{Quad}\left(A, \mathbf{k}^\times\right) \ . \tag{4.18}$$

Cocycle representatives in $H^3_{\mathrm{ab}}(A, \mathbf{k}^\times)$ are pairs (τ, b) where τ is an ordinary group cocycle on A, and b is a map $A^2 \to \mathbf{k}^\times$ satisfying the two hexagon equations with τ [JS93] (see [EGNO15, §8.4] for explicit formulas). Explicitly, the identification (4.18) sends a pair (τ, b) to the quadratic form defined by $q(a) \coloneqq b(a, a)$. Given such a pair (τ, b), we obtain a not necessarily symmetric braided structure on the monoidal category of A-graded vector spaces with associator twisted by τ. We can specify this braiding on simple objects:

$$\mathbf{k}_a * \mathbf{k}_b = \mathbf{k}_{a+b} \xrightarrow[\sim]{b(a,b)\cdot \mathrm{id}_{\mathbf{k}_{a+b}}} \mathbf{k}_{a+b} = \mathbf{k}_b * \mathbf{k}_a \ . \tag{4.19}$$

Example 4.20. This is a continuation of Example 4.17. The pointed braided fusion category $\mathcal{Z}(\mathbf{Vect}[L])$ is classified by the finite abelian group $A = L \oplus L^*$ with quadratic form $q = \mathrm{ev}$ (see (4.25) below).

A degree-3 cocycle in abelian group cohomology is a pair (τ, b) where τ is an ordinary group 3-cocycle on A and $b\colon A^2 \to \mathbf{k}^\times$ satisfying the hexagon equations with τ, as above.

The abelian group 3-cocycle classifying $\mathcal{Z}(\mathbf{Vect}[L])$ has trivial τ, and $b = b_0$ is a map from A^2 to $\mathbf{k}^\times$ such that it agrees with q when restricted to the diagonal, i.e. for all $a \in A$:

$$b_0(a, a) = \mathrm{ev}(a) \ .$$

For example, b_0 might be taken to be:

$$b_0((\ell_1, \chi_1), (\ell_2, \chi_2)) = \chi_1(\ell_2) \ .$$

Other choices for b_0 differ from this by an abelian group coboundary in degree 3.

4.3.3. *Higher groupoids in the pointed case.* When the (braided) fusion categories in question are pointed, the groupoids introduced in Section 4.2 have a more concrete description. Let (A, q) be a metric group, and let $\mathcal{A} = \mathbf{Vect}[A]$ be the associated braided category of A-graded vector spaces. The Picard 3-type of $\mathcal{A}$, i.e. the classifying space $B\,\mathrm{Pic}(\mathcal{A})$ of the Picard 3-group from Section 4.2, has the following homotopy groups:

$$\pi_1 = \mathrm{O}(A, q) \qquad \pi_2 = A \qquad \pi_3 = \mathbf{k}^\times \ . \tag{4.21}$$

Remark 4.22. Following [ENO10, Remark 10.7], these homotopy groups appear as follows. First of all, we know that $\pi_2 = A$ and $\pi_3 = \mathbf{k}^\times$, since this is the underlying groupoid of $\mathcal{A}$ itself (delooped twice).

Now we identify π_1. The Whitehead half-square (see [ENO10, §7.3]) is a homotopy invariant, which is a map $\pi_2 \to \pi_3$. By [ENO10, Proposition 7.6], it is given by the self-braiding on invertible objects of $\mathcal{A}$, which is q in this case. The action of π_1 on π_2 by conjugation must preserve this invariant, and hence this defines a map $\pi_1 \to \mathrm{O}(A, q)$, which turns out to be an equivalence.

This conjugation action is a shadow of a more general way to understand these groupoids: The invertible part of any monoidal 2-category acts on the endomorphisms of the identity object by conjugation. As is explained in [ENO10, Remark 5.4], by setting the monoidal 2-category to be $\mathrm{End}_{\mathbf{Fus}}(\mathcal{C})$, we obtain a map:

$$\mathrm{Aut}_{\mathbf{Fus}}(\mathcal{C}) \to \mathrm{Aut}_{\mathbf{EqBr}}(\mathcal{Z}(\mathcal{C})) \ .$$

When we restrict this to the truncation of $\mathrm{Aut}_{\mathbf{Fus}}(\mathcal{C})$ to a 1-category (i.e. 2-group), we obtain the equivalence (4.8). When we set the 2-category to be the category of modules over $\mathcal{Z}(\mathcal{C})$, we obtain the equivalence (4.4).

4.3.4. *Lagrangian subgroups.*

Definition 4.23. A subgroup $L \subset A$ of a metric group (A, q) is *isotropic* if $q(\ell) = 1$ for all $\ell \in L$. The subgroup is *Lagrangian* if it is isotropic and $|L|^2 = |A|$.

Example 4.24. If $A = L \oplus L^*$ and $q = \mathrm{ev}$, then $L \oplus \{0\}$ and $\{0\} \oplus L^*$ are both Lagrangian subgroups.

Let L be a finite abelian group. The Drinfeld center of the fusion category $(\mathbf{Vect}[L], *)$ is:

$$\mathcal{Z}(\mathbf{Vect}[L], *) \cong \mathbf{Vect}[L \oplus L^*] \ . \tag{4.25}$$

Indeed, a simple object of the center consists of a simple object $\mathbf{k}_\ell$, $\ell \in L$, together with a half-braiding, which is given by a character $\chi \in L^*$. The tensor structure is convolution, and the braiding is induced by evaluation, as in Example 4.17.

Recall from Section 4.3.2, that pointed braided fusion categories are classified by finite abelian groups equipped with a quadratic form. In this case, the finite abelian group associated to $\mathcal{Z}(\mathbf{Vect}[L])$ is $L \oplus L^*$, and the quadratic form is $\mathrm{ev}\colon L \oplus L^* \to \mathbf{k}^\times$. Lagrangians in the metric group give rise to fusion categories $\mathcal{C}$ such that $\mathcal{A} \cong \mathcal{Z}(\mathcal{C})$.

4.3.5. *Pointed Drinfeld centers.* Let $\mathcal{C} = \mathbf{Vect}[L]^\tau$ for L a finite group and τ a 3-cocycle on L. Following [MN18, §4], for any ℓ in the center of L, we can define:

$$T_\ell(x, y) = \frac{\tau(\ell, x, y)\,\tau(x, y, \ell)}{\tau(x, \ell, y)} \ . \tag{4.26}$$

As is shown in [MN18, §4.1], T_ℓ is a 2-cocycle on the group L, and $T_{(-)}$ defines a group homomorphism from the center of L to $H^2(L, \mathbf{k}^\times)$. Then, as in [MN18, Corollary 4.3], $\mathcal{Z}(\mathcal{C})$ is pointed if and only if L is abelian and $T_{(-)}$ is the trivial group homomorphism.

Example 4.27. When L is a vector space over $\mathbb{F}_p$ with p odd, by [MN18, Corollary 5.3] $\mathcal{Z}(\mathcal{C})$ is pointed if and only if τ has trivial alternating component. (Recall $H^3(L, \mathbf{k}^\times)$ decomposes as a direct sum of the alternating factor $\wedge^3 L^*$ and the symmetric factor $\mathrm{Sym}^2 L^*$.)

4.3.6. *Polarizability.*

Definition 4.28. A pointed braided fusion category is *polarizable* if it is the Drinfeld center of another fusion category $\mathcal{C}$.

The usual classification of pointed braided fusion categories (Section 4.3.2) restricts to a classification of polarizable pointed braided fusion categories as follows.

Proposition 4.29. *A pointed braided fusion category classified by (A, q) is polarizable if and only if q is nondegenerate and A contains a Lagrangian subgroup L* [DGNO10, §4.4, in particular Theorem 4.64 and Remark 4.67(i)] *(for the "only if" direction when $\mathcal{C}$ is not assumed to be pointed: if the*

category classified by (A,q) is a Drinfeld center, then it is trivial in the Witt group [DMNO13, Proposition 5.8]*, hence (A,q) is trivial in the Witt group of metric groups* [DMNO13, Proposition 5.17]*, i.e. some isotropic subgroup $H \subset A$ satisfies $H^{\perp}/H = 0$, so H is Lagrangian). In this case $\mathcal{C}$ can be taken to be pointed, i.e. of the form $\mathbf{Vect}\,[L^*]^{\tau}$, and A is an extension:*

$$0 \to L \to A \to L^* \to 0$$

where the second map is induced by $\langle \cdot, \cdot \rangle_q$.

Definition 4.30. A *polarized metric group* is a triple (A,q,L) consisting of a metric group (A,q) and a Lagrangian subgroup $L \subset A$.

4.4. **Obstruction theory.** We give an executive summary of the obstruction theory developed in [ENO10].

4.4.1. *Postnikov and Whitehead towers.* Recall the definition of the Postnikov and Whitehead towers of a space, and the associated k-invariants. See [Pos55, Whi78], or [MP12, Chapter 3] for references.

Using the identifications (4.6) and (4.4), the 3-type $B\operatorname{Aut}_{\mathbf{Fus}}(\mathcal{C}) \simeq B\operatorname{Pic}(\mathcal{Z}(\mathcal{C}))$ has Postnikov and Whitehead towers:

$$\begin{array}{ccccc} * & \longrightarrow & B^3\pi_3 & \longrightarrow & B^2\mathcal{Z}(\mathcal{C})^{\times} \longrightarrow B\operatorname{Aut}_{\mathbf{Fus}}(\mathcal{C}) \\ \downarrow & & \downarrow & & \downarrow \\ * & \longrightarrow & B^2\pi_2 & \longrightarrow & B\operatorname{Aut}_{\mathbf{EqBr}}(\mathcal{Z}(\mathcal{C})) \\ & & \downarrow & & \downarrow \\ & & * & \longrightarrow & B\pi_1 \\ & & & & \downarrow \\ & & & & * \end{array} \tag{4.31}$$

Recall the homotopy groups π_i from (4.21) and Remark 4.22. The 1-connected cover of $B\operatorname{Aut}_{\mathbf{Fus}}(\mathcal{C}) \simeq B\operatorname{Pic}(\mathcal{Z}(\mathcal{C}))$ is the classifying space of the connected component of the identity in $\operatorname{Pic}(\mathcal{Z}(\mathcal{C}))$, i.e. of the regular module $\mathcal{Z}(\mathcal{C})_{\mathcal{Z}(\mathcal{C})}$. The automorphisms of the regular module are the invertible objects of $\operatorname{End}_{\mathcal{Z}(\mathcal{C})}(\mathcal{Z}(\mathcal{C})) \simeq \mathcal{Z}(\mathcal{C})$, so this cover is the underlying groupoid of $\mathcal{Z}(\mathcal{C})^{\times}$ delooped twice, $B^2\mathcal{Z}(\mathcal{C})^{\times}$ (as in Remark 4.22). See Remark 5.2 for more on the space $B^2\mathcal{Z}(\mathcal{C})^{\times}$.

4.4.2. *Degree three obstruction.* Let $\mathcal{A}$ be a nondegenerate braided fusion category, e.g. $\mathcal{A} = \mathcal{Z}(\mathcal{C})$.

Question 3. Given a group G and a morphism of groups $f\colon G \to \pi_1$, what is the obstruction to performing the following lift?

$$\begin{array}{ccc} & & \operatorname{Aut}_{\mathbf{EqBr}}(\mathcal{A}) \\ & \overset{\exists}{\nearrow} & \downarrow \\ G & \xrightarrow{f} & \pi_1 \end{array}$$

This question is equivalent to asking if the k-invariant $k_2 \in H^3(B\pi_1, \pi_2)$ of the 2-type $B\operatorname{Aut}_{\mathbf{EqBr}}(\mathcal{A})$ pulls back to something trivializable on G. I.e. the pullback

$$O_3(f) \coloneqq (Bf)^* k_2 \in H^3(G, \pi_2) \tag{4.32}$$

is the obstruction to lifting:

$$\begin{array}{ccc} & & B\operatorname{Aut}_{\mathbf{EqBr}}(\mathcal{A}) \\ & \overset{\exists}{\nearrow} & \downarrow \\ BG & \xrightarrow[Bf]{} & B\pi_1 = \pi_{\leq 1} B\operatorname{Aut}_{\mathbf{EqBr}}(\mathcal{A}) \end{array}$$

Remark 4.33. At the level of classifying spaces, (4.4) tells us that the truncation of $B\operatorname{Pic}(\mathcal{A})$ to a 2-type is identified with $B\operatorname{Aut}_{\mathbf{EqBr}}(\mathcal{A})$. Therefore O_3 is the same as the obstruction to lifting:

$$\begin{array}{ccccc} & & \pi_{\leq 2}B\operatorname{Pic}(\mathcal{A}) & & \\ & \nearrow & & \searrow & \\ BG & \xrightarrow{Bf} & B\pi_1 & \simeq & \pi_{\leq 1}B\operatorname{Pic}(\mathcal{A}) \end{array}$$

Remark 4.34. When $\mathcal{A} = \mathcal{Z}(\mathcal{C})$ for some fusion category $\mathcal{C}$, (4.6) allows us to describe this obstruction in terms of the Brauer-Picard 3-type $B\operatorname{Aut}_{\mathbf{Fus}}(\mathcal{C})$. In particular, it is the same as the obstruction to lifting:

$$\begin{array}{ccccc} & & \pi_{\leq 2}B\operatorname{Aut}_{\mathbf{Fus}}(\mathcal{C}) & & \\ & \nearrow & & \searrow & \\ BG & \xrightarrow{Bf} & B\pi_1 & \simeq & \pi_{\leq 1}B\operatorname{Aut}_{\mathbf{Fus}}(\mathcal{C}) \end{array}$$

4.4.3. *Degree four obstruction in the split case.* Assume that the obstruction $O_3(f)$ is trivializable, and let s be the lift of $Bf\colon BG \to B\pi_1$ to $B\operatorname{Aut}_{\mathbf{EqBr}}(\mathcal{A})$ corresponding to a fixed trivialization of $O_3(f)$. Recall the trivializations of $O_3(f)$ form a torsor over $H^2(BG, \pi_2)$. By Remark 4.33, the upshot of the existence of s is that we have an associated map $Bf_s : BG \to \pi_{\leq 2}B\operatorname{Pic}(\mathcal{A})$.

Question 4. What is the obstruction to performing the following lift?

$$\begin{array}{ccc} & & B\operatorname{Pic}(\mathcal{A}) \\ & \nearrow & \downarrow \\ BG & \xrightarrow{Bf_s} & \pi_{\leq 2}B\operatorname{Pic}(\mathcal{A}) \end{array}$$

Just as before, the obstruction comes from the k-invariant. Now the relevant k-invariant is

$$k_3 \in H^4(\pi_{\leq 2}B\operatorname{Pic}(\mathcal{A}), \pi_3) \ . \tag{4.35}$$

The pullback

$$O_4(f, s) \coloneqq (Bf_s)^* k_3 \in H^4(BG, \pi_3)$$

is the obstruction to lifting.

Remark 4.36. Again, if $\mathcal{A} \cong \mathcal{Z}(\mathcal{C})$ for some fusion category $\mathcal{C}$ then, by Remark 4.34, we have an associated map $Bf_s : BG \to \pi_{\leq 2}B\operatorname{Aut}_{\mathbf{Fus}}(\mathcal{C})$. By (4.6), the above question is equivalent to the following.

Question 5. What is the obstruction to performing the following lift?

$$\begin{array}{ccc} & & B\operatorname{Aut}_{\mathbf{Fus}}(\mathcal{C}) \\ & \nearrow & \downarrow \\ BG & \xrightarrow{Bf_s} & \pi_{\leq 2}B\operatorname{Aut}_{\mathbf{Fus}}(\mathcal{C}) \end{array}$$

The obstruction to lifting this is still $O_4(f, s) \in H^4(BG, \pi_3)$.

As it turns out, e.g. from [ENO10, Proposition 7.2, 7.3]:

$$\pi_3(B\operatorname{Aut}_{\mathbf{Fus}}(\mathcal{C})) \cong \mathbf{k}^\times \cong \pi_3(B\operatorname{Pic}(\mathcal{A})) \ .$$

Therefore the obstruction is classified by familiar $\mathbf{k}^\times$-cohomology:

$$O_4(f, s) \in H^4\left(BG, \mathbf{k}^\times\right) \ . \tag{4.37}$$

4.4.4. *Full degree four obstruction.* In Section 4.4.3, we assume that the map into $\pi_{\leq 2} B\operatorname{Pic}(\mathcal{A}) \simeq B\operatorname{Aut}_{\mathbf{EqBr}}(\mathcal{A})$ is given by a trivialization of $O_3(f)$ for f some map from a group to π_1. There is a slightly more general obstruction one can consider.

Question 6. Given a 2-group G_2, and a map

$$f_2 \colon BG_2 \to \pi_{\leq 2} B\operatorname{Pic}(\mathcal{A}) \simeq B\operatorname{Aut}_{\mathbf{EqBr}}(\mathcal{A}) \ ,$$

what is the obstruction to performing the following lift?

$$\begin{array}{ccc} & & B\operatorname{Pic}(\mathcal{A}) \\ & \overset{}{\nearrow} & \big\downarrow \\ BG_2 & \xrightarrow{f_2} & \pi_{\leq 2} B\operatorname{Pic}(\mathcal{A}) \end{array}$$

This is equivalent to asking if the k-invariant k_3 from (4.35) pulls back to something trivializable on G_2.

Remark 4.38. As in the preceding subsections, this is equivalent to the analogous lifting question for $B\operatorname{Aut}_{\mathbf{Fus}}(\mathcal{C})$, by (4.6). Furthermore, a combination of (4.4) and (4.6) tell us that we can consider the universal version of this question: $G_2 = \operatorname{Aut}_{\mathbf{EqBr}}(\mathcal{Z}(\mathcal{C}))$. I.e.

$$k_3 \colon B\operatorname{Aut}_{\mathbf{EqBr}}(\mathcal{Z}(\mathcal{C})) \to B^4\mathbf{k}^\times \tag{4.39}$$

is the obstruction to lifting

$$\begin{array}{ccc} & & B\operatorname{Aut}_{\mathbf{Fus}}(\mathcal{C}) \\ & \overset{}{\nearrow} & \big\downarrow \\ B\operatorname{Aut}_{\mathbf{EqBr}}(\mathcal{Z}(\mathcal{C})) & \xrightarrow{\sim} & \pi_{\leq 2} B\operatorname{Aut}_{\mathbf{Fus}}(\mathcal{C}) \end{array}$$

4.4.5. *Obstruction theory in the pointed case.* Recall the homotopy groups in (4.21). If we take $G = \mathrm{O}(A, q)$, we get a universal obstruction class:

$$O_3 \in H^3(B\,\mathrm{O}(A, q), A) \ .$$

Let $|A|$ be odd. Then, as discussed in [EG18, §6] (and [CGPW16, §5]), the obstruction O_3 vanishes, and in fact has a canonical splitting:

$$s \colon B\,\mathrm{O}(A, q) \to \pi_{\leq 2} B\operatorname{Pic}(\mathcal{A}) \ .$$

Therefore we have a well-defined universal version of the obstruction class in (4.37):

$$O_4(A, q) \in H^4\left(B\,\mathrm{O}(A, q), \mathbf{k}^\times\right) \ . \tag{4.40}$$

Remark 4.41. In other words, $s^* B\operatorname{Pic}(\mathcal{A})$ (which is $s^* B\operatorname{Aut}_{\mathbf{Fus}}(\mathcal{C})$ when $\mathcal{A} = \mathcal{Z}(\mathcal{C})$) defines a $B^3\mathbf{k}^\times$-bundle over $B\,\mathrm{O}(A, q)$. At the level of 3-groups, this is actually a *central* extension of $\mathrm{O}(A, q)$ by $B^2\mathbf{k}^\times$ since π_1 acts on $\pi_3 = \mathbf{k}^\times$ by conjugation, and conjugating a scalar automorphism by a $\mathbf{k}$-linear equivalence returns the same scalar.

Example 4.42. Let $G = \mathbb{Z}/2$. Then $H^4(BG, \mathbf{k}^\times) = 0$ so the class O_4 necessarily vanishes, and so the anomaly vanishes. There are two nonequivalent trivializations classified by $H^3(BG, \mathbf{k}^\times) = \mathbb{Z}/2$.

This is used in [ENO10, Example 9.4] to reproduce the classification in [TY98] of $\mathbb{Z}/2$-graded fusion categories.

Remark 4.43. Recall that the trivializations of $O_3(f)$ form a torsor over $H^2(BG, \pi_2)$. I.e. given two trivializations s and s' of $O_3(f)$, there exists $L \in H^2(BG, \pi_2)$ such that $s' = Ls$. By [ENO10, Proposition 8.15] we have that

$$O_4(f, s')\,/O_4(f, s) = \mathrm{PW}(L) \ ,$$

where PW denotes the *Pontrjagin-Whitehead quadratic function* from [ENO10, §8.7].

Also see [Jen11, Proposition 7.3] and [CGPW16, Proposition 8] for a concrete formula for the Pontrjagin-Whitehead quadratic function. It is shown to vanish in some examples in [GJ19].

4.4.6. *Vanishing of the obstruction over a finite field.* Let p be an odd prime, and let $A = V$ be a $2n$-dimensional vector space over the finite field $\mathbb{F}_p$. Let q_{split} be the quadratic form on V of signature (n, n).

Observation 4.44. *The orthogonal group* $\mathrm{O}(V, q_{\text{split}})$ *is the split orthogonal group over* $\mathbb{F}_p$.

Note that, by (4.25), if L is a vector space over $\mathbb{F}_p$ of dimension n, then

$$\mathcal{Z}(\mathbf{Vect}[L], *) \simeq (\mathbf{Vect}[V], *, \beta_q) \ ,$$

where β_q is the braiding (4.19) given by an abelian 3-cocycle representing $q = q_{\text{split}}$. The obstruction $O_4(A, q_{\text{split}})$ vanishes in this case by [EG18, Theorem 6.1].

Recall the analogy between this categorical representation of $\mathrm{O}(V, q)$ and the Weil representation of the metaplectic group in Remark 4.58. This vanishing can be thought as an analogue of the fact that the Weil representation splits over a finite field [GH09, GH11].

4.4.7. *Braided categories with a prescribed obstruction class.*

Proposition 4.45. *For any finite group G and group 4-cocycle π there exists a nondegenerate braided fusion category $\mathcal{B}$ and monoidal functor*

$$\Phi\colon G \to \mathrm{Aut}_{\mathbf{EqBr}}(\mathcal{B})$$

such that $[\pi] = (B\Phi)^* k_3$, *with* k_3 *the k-invariant* (4.35) *of* $\pi_{\leq 2} B\,\mathrm{Pic}(\mathcal{B})$ *(Section 4.4.3 with* $\mathcal{A} = \mathcal{B}$*).*

Proof. Recall that every fusion 2-category is Morita-equivalent to a connected fusion 2-category [Déc24, Theorem 4.2.2]. Therefore the fusion 2-category $2\mathbf{Vect}[G]^{\pi}$ is Morita equivalent to the 2-category of module categories over some braided fusion category $\mathcal{B}$.

In fact, we obtain more: As a special case of the classification of fusion 2-categories [DHJF$^+$24], the connected ones are classified by a nondegenerate braided fusion category $\mathcal{B}$ equipped with a monoidal functor $\Phi\colon G \to \mathrm{Aut}_{\mathbf{EqBr}}(\mathcal{B})$. The fact that $(B\Phi)^* k_3 = [\pi]$ follows from the reconstruction of a connected fusion 2-category from $\mathcal{B}$ and Φ, as in [DHJF$^+$24, §4.4]. □

4.5. **Extensions of the finite orthogonal group.** Throughout this section we assume $|A|$ is odd, as in Section 4.4.5. Recall, from Remark 4.41, that we can interpret $O_4 = O_4(A, q)$ (from Section 4.4.5) as encoding a central extension of $B\,\mathrm{O}(A, q)$ by $B^3\mathbf{k}^\times$. In particular, $\mathrm{hofib}(O_4)$ is a $B^3\mathbf{k}^\times$-bundle over $B\,\mathrm{O}(A, q)$:

$$\begin{array}{ccc} B^3\mathbf{k}^\times & \longrightarrow & \mathrm{hofib}(O_4) \\ \downarrow & & \downarrow \\ * & \longrightarrow & B\,\mathrm{O}(A, q) \end{array}$$

Definition 4.46. Let $\mathrm{3Lip}(A, q)$ denote the 3-group $\Omega\,\mathrm{hofib}(O_4)$, i.e. the 3-group with classifying space $\mathrm{hofib}(O_4)$.

Remark 4.47. The name comes from the analogy in Section 4.7. Namely, $\mathrm{3Lip}(A, q)$ is a 3-group analogue of the Lipschitz group (a.k.a. Clifford group) of a quadratic vector space.

Assume moreover that (A, q) is polarized by a Lagrangian $L \subset A$ (see Proposition 4.29), and let $l = |L|$. The order of the class $O_4(A, q)$ is shown in [ENO10, Theorem 8.16] to divide l^4. In other words, it is the image of some class $c \in H^4(B\,\mathrm{O}(A, q), \mu_{l^4})$. This class is only well-defined up to the image of the Bockstein $H^3(B\,\mathrm{O}(A, q), \mathbf{k}^\times) \to H^4(B\,\mathrm{O}(A, q), \mu_{l^4})$, and we fix such a choice. In particular $\mathrm{3Pin}(A, q)$ and $\mathrm{3Spin}(A, q)$ depend on this choice.

Definition 4.48. Define the 3-group $\mathrm{3Pin}(A, q)$ as $\Omega\,\mathrm{hofib}(c)$.

We will henceforth write this class c as:

$$c\left(3\mathrm{Pin}\right) \in H^4\left(B\,\mathrm{O}\left(A,q\right), \mu_{l^4}\right) \ . \tag{4.49}$$

By definition, $c\left(3\mathrm{Pin}\right)$ classifies $B\,3\mathrm{Pin}$ over $B\,\mathrm{O}\left(A,q\right)$. We have a diagram

$$\begin{array}{ccccccccc} & & & & 1 & & 1 & & \\ & & & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & B^3\mu_{l^4} & \longrightarrow & B\,3\mathrm{Pin}\left(A,q\right) & \longrightarrow & B\,\mathrm{O}\left(A,q\right) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & B^3\mathbf{k}^\times & \longrightarrow & B\,3\mathrm{Lip}\left(A,q\right) & \longrightarrow & B\,\mathrm{O}\left(A,q\right) & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle B^3(-)^{l^4}} & & \downarrow{\scriptstyle N_{(A,q)}} & & & & \\ & & B^3\mathbf{k}^\times & = & B^3\mathbf{k}^\times & & & & \end{array} \tag{4.50}$$

where the rows and the first two columns are fiber sequences. Here $N_{(A,q)}$ is the composite

$$\mathrm{hofib}\left(O_4\right) \to \mathrm{hofib}\left(\left(-\right)^{l^4} \circ O_4\right) \simeq B\,\mathrm{O}\left(A,q\right) \times B^3\mathbf{k}^\times \to B^3\mathbf{k}^\times \ ,$$

where the equivalence uses the null-homotopy of $\left(-\right)^{l^4} \circ O_4 = \left(-\right)^{l^4} \circ \iota \circ c\left(3\mathrm{Pin}\right)$ given by the exact sequence $\mu_{l^4} \xrightarrow{\iota} \mathbf{k}^\times \xrightarrow{\left(-\right)^{l^4}} \mathbf{k}^\times$. On π_3 it sends a scalar to its l^4-th power, its homotopy fiber is $B\,3\mathrm{Pin}\left(A,q\right)$, and $N_{(A,q)}$ depends on the choice of $c\left(3\mathrm{Pin}\right)$ made above.

Recall $\mathrm{SO}\left(A,q\right)$ from (4.16). The class $c\left(3\mathrm{Pin}\right)$ in (4.49) pulls back to a class on $\mathrm{SO}\left(A,q\right)$, which we suggestively write as

$$c\left(3\mathrm{Spin}\right) \in H^4\left(B\,\mathrm{SO}\left(A,q\right), \mu_{l^4}\right) \ . \tag{4.51}$$

Definition 4.52. Define the 3-group $3\mathrm{Spin}\left(A,q\right)$ as $\Omega\,\mathrm{hofib}\left(c\left(3\mathrm{Spin}\right)\right)$.

The following follows from the definition of 3Pin, 3Spin, and the universal property of the homotopy fiber.

Proposition 4.53. *The* 3*-group* $3\mathrm{Spin}\left(A,q\right)$ *is the pullback:*

$$\begin{array}{ccc} 3\mathrm{Spin}\left(A,q\right) & \dashrightarrow & 3\mathrm{Pin}\left(A,q\right) \\ \vdots\downarrow & & \downarrow \\ \mathrm{SO}\left(A,q\right) & \longrightarrow & \mathrm{O}\left(A,q\right) \end{array}$$

Remark 4.54. Note that if $O_4\left(A,q\right)$ vanishes, then Definitions 4.46, 4.48, and 4.52 are trivial (i.e. split) extensions of $\mathrm{O}\left(A,q\right)$ (resp. $\mathrm{SO}\left(A,q\right)$). E.g. if A is a vector space over $\mathbb{F}_p$ and q is the split quadratic form as discussed in Section 4.4.6.

4.6. **Interlude: spinors.** We collect some well-known facts about the Clifford algebra and spinors. We refer the reader to [Del99] for a more in-depth reference.

Let $\left(V,q\right)$ be a nondegenerate quadratic vector space over a field $\mathbf{k}$ with $\mathrm{char}\,\mathbf{k} \neq 2$. The *Clifford algebra* is the following quotient of the tensor algebra:

$$\mathrm{Cliff}\left(V,q\right) \coloneqq T\left(V\right) / \left(v \otimes v = q\left(v\right)\right) \ .$$

One can calculate the anti-commutators in $\mathrm{Cliff}\left(V,q\right)$ to be:

$$uv + vu = b_q\left(u,v\right),$$

where b_q is the symmetric bihomomorphism corresponding to q (recall $\operatorname{char} k \neq 2$). Note that $\operatorname{Cliff}(V,q)$ is $\mathbb{Z}/2$-graded, with the image of V being odd. Write $p(a)$ for the parity of a homogeneous element. The tensor algebra has an anti-automorphism sending

$$v_1 \otimes \cdots \otimes v_N \mapsto v_N \otimes \cdots \otimes v_1$$

in degree N. The ideal we quotient out by to define Cliff is preserved by this anti-automorphism, so we obtain an antiautomorphism of Cliff. Write $a \mapsto a^T$ for this antiautomorphism.

Define the group Γ to be the subgroup of $\operatorname{Cliff}(V,q)^\times$ consisting of homogeneous elements which normalize the copy of V inside of $\operatorname{Cliff}(V,q)$. This is sometimes called the *Clifford group* or the *Lipschitz group*. Let $g \in \Gamma$ act on V by sending:

$$v \mapsto (-1)^{p(g)} gvg^{-1} \ .$$

This defines a map $\Gamma \to \mathrm{O}(V,q)$. Since $\operatorname{char} k \neq 2$, $\mathrm{O}(V,q)$ is generated by reflections (by the Cartan–Dieudonné theorem) so this map is onto. The kernel is given by scalars, so we have a short exact sequence

$$1 \to \mathbf{k}^\times \to \Gamma \to \mathrm{O}(V,q) \to 1 \ .$$

We cut the coefficients down to $\{\pm 1\} \hookrightarrow \mathbf{k}^\times$ as follows. The spinor norm is the map

$$\begin{aligned} \Gamma &\xrightarrow{N} \mathbf{k}^\times \\ g &\longmapsto gg^T \end{aligned}$$

Define the group:

$$\operatorname{Pin}(V,q) \coloneqq \ker(N) \ .$$

There is a unique map $N_O \colon \mathrm{O}(V) \to \mathbf{k}^\times / (\mathbf{k}^\times)^2$, which is also sometimes called the spinor norm, such that

$$\begin{array}{ccc} \Gamma & \longrightarrow & \mathrm{O}(V) \\ \downarrow{\scriptstyle N} & & \downarrow{\scriptstyle N_O} \\ \mathbf{k}^\times & \longrightarrow & \mathbf{k}^\times / (\mathbf{k}^\times)^2 \end{array}$$

This fits into the following diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \{\pm 1\} & \longrightarrow & \operatorname{Pin}(V,q) & \longrightarrow & \ker(N_O) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathbf{k}^\times & \longrightarrow & \Gamma & \longrightarrow & \mathrm{O}(V) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow{\scriptstyle N} & & \downarrow{\scriptstyle N_O} & & \\ 1 & \longrightarrow & (\mathbf{k}^\times)^2 & \longrightarrow & \mathbf{k}^\times & \longrightarrow & \mathbf{k}^\times / (\mathbf{k}^\times)^2 & \longrightarrow & 1 \end{array} \tag{4.55}$$

where the rows are short exact sequences. The spinor norm N_O is the obstruction to $\operatorname{Pin}(V)$ being a double cover of $\mathrm{O}(V)$: $\operatorname{Pin}(V)$ is automatically a double cover of $\ker(N_O)$, but if N_O is trivial then $\ker(N_O) = \mathrm{O}(V)$.

Example 4.56. If $\mathbf{k}$ is algebraically closed then $\mathbf{k}^\times / (\mathbf{k}^\times)^2$ is trivial, so the spinor norm is necessarily trivial. If $\mathbf{k} = \mathbb{R}$, then the spinor norm vanishes if q is positive definite. Therefore $\operatorname{Pin}(V)$ is a double cover of $\mathrm{O}(V)$ in these cases.

Define Spin to be the restriction of this extension to $\mathrm{SO} \subset \mathrm{O}(V)$. I.e. the following pullback.

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \{\pm 1\} & \longrightarrow & \operatorname{Pin}(V) & \longrightarrow & \mathrm{O}(V) & \longrightarrow & 1 \\ & & \| & & \uparrow & & \uparrow & & \\ 1 & \longrightarrow & \{\pm 1\} & \longrightarrow & \operatorname{Spin}(V) & \longrightarrow & \mathrm{SO}(V) & \longrightarrow & 1 \end{array}$$

For ease of exposition, assume $\dim V = 2n$ and $\mathbf{k} = \mathbb{C}$ (or $\mathbf{k} = \mathbb{R}$ with q of signature (n,n)). In this case the algebra $\mathrm{Cliff}(V,q)$ is the matrix algebra $M_{2^n}(\mathbf{k})$, and therefore has a unique simple module S up to isomorphism. Note that the automorphism group of the module S is $\mathbf{k}^\times$. Automorphisms of $\mathrm{Cliff}(V)$ only lift to automorphisms of the module S up to isomorphism, i.e. up to scalars. In particular this makes S a projective $\mathrm{O}(V,q)$-representation.

There exists a Lagrangian subspace $L \subset V$ and an identification $V \simeq L \oplus L^*$, and a model for the simple module S is given by the exterior algebra $\wedge^\bullet L^*$. The algebra $\mathrm{Cliff}(V)$ can be identified with $\mathrm{End}(\wedge^\bullet L^*)$ by sending $(\ell, 0) \in V$ to $\ell \wedge (-)$, and $(0,\varphi) \in V$ to the interior product ι_φ.

4.7. **A detailed analogy with** Spin**.** See Section 4.6 where we collect some basic facts and notation involving the well-known spin-representation. In Table 1, we present an analogy between various objects in the theory of the spin-representation, and various objects introduced throughout Section 4.

The Clifford algebra associated to a quadratic vector space (V,q) is analogous to the braided fusion category associated to a metric group (A,q). One should think that this is a double-categorification: $\mathbb{E}_1$-algebras are being replaced with $\mathbb{E}_2$-categories. The Clifford algebra can be viewed as an associative deformation of the exterior algebra: the anticommutator in the Clifford algebra $\{x,y\} = q(x,y)$ becomes the usual multiplication on the exterior algebra (up to sign) for trivial q. Similarly, we can think that the braided fusion category $\mathcal{A}$ is a braided deformation of the symmetric fusion category $\mathbf{Vect}[A]$ with convolution: The braiding

$$\beta_q \colon \mathbf{k}_a * \mathbf{k}_b \xrightarrow{b(a,b)\,\mathrm{id}} \mathbf{k}_b * \mathbf{k}_a$$

becomes the symmetric braiding for trivial q.

The group $\mathrm{O}(V,q)$ plays the same role as $\mathrm{O}(A,q)$, and they both contain their respective special orthogonal subgroups as the kernel of the determinant. The ordinary group of scalars $\mathbf{k}^\times$ is replaced by the 3-group $B^2\mathbf{k}^\times$. Again we have gone up two categorical levels from an ordinary group to a 3-group. The group of automorphisms of $\mathrm{Cliff}(V)$ contains the affine orthogonal group $V \rtimes \mathrm{O}(V,q)$. As long as A is of odd order, the entire 2-group $\mathrm{Aut}_{\mathbf{EqBr}}(\mathcal{A})$ is a semidirect product of BA and $\mathrm{O}(A,q)$. This is one instance where things become easier upon categorification. See Remark 4.58.

The Clifford/Lipschitz group $\Gamma = \mathrm{Lip}(V,q)$ is a central extension of $\mathrm{O}(V,q)$ by $\mathbf{k}^\times$, just as $3\mathrm{Lip}(A,q)$ is a central extension of $\mathrm{O}(A,q)$ by $B^2\mathbf{k}^\times$ (Remark 4.41). The cohomology class $O_4(A,q)$ is of order dividing l^4 (where $l = |L| = \sqrt{|A|}$), whereas Γ is induced by a double cover. I.e. $B^2\mu_{l^4}$ plays the role of μ_2. The 3-group 3Pin (resp. 3Spin) is the analogue of Pin (resp. Spin). This is a difference between the two sides of the analogy: 3Pin and 3Spin are not higher double covers (i.e. extensions by $B^2\mu_2$) even though Pin and Spin are double covers. Note that the spinor norm N in (4.55) is analogous to $N_{(A,q)}$ in (4.50).

Remark 4.57. Another aspect of the spin group is that (in signature (m,n) with either m or n being ≤ 1) it is the universal cover of SO. However, outside of these cases the Spin groups have fundamental group $\mathbb{Z}/2$. In particular, the split case $\mathrm{Spin}(n,n)$ is not simply-connected, and this is the case which is most directly analogous to the 3-dimensional setting: $q = \mathrm{ev}$ (for A a vector space over $\mathbb{F}_p$) has signature (n,n).

The module $S = \wedge^\bullet L^*$ is analogous to the fusion category $\mathcal{C} = \mathbf{Vect}[L^*]$ with convolution. This is again a double-categorification: We have replaced $\mathbb{E}_0$-algebras with $\mathbb{E}_1$-categories. Just as S is a module over $\mathrm{Cliff}(V)$, $\mathcal{C}$ is a module over $\mathcal{A}$, since $\mathcal{A} \cong \mathcal{Z}(\mathcal{C})$.

The 3-group $\mathrm{Pic}(\mathcal{A})$ is identified with invertible $\mathcal{C}$-bimodules,

$$\mathrm{Pic}(\mathcal{A}) \simeq \mathrm{Aut}_{\mathbf{Fus}}(\mathcal{C}) \ ,$$

in an analogous way in which the Clifford algebra itself is identified with the endomorphisms of spinors. Therefore Γ (resp. Spin) acts on S just as 3Lip (resp. 3Spin) automatically acts on $\mathcal{C}$.

Remark 4.58. This suggests that the analogue of $\mathrm{Pic}(\mathcal{A})$ should be the invertible part of the Clifford algebra. This is not entirely the case: an important fact about $\mathrm{Pic}(\mathcal{A})$ is that it is an extension of $\mathrm{Aut}_{\mathbf{EqBr}}(\mathcal{A})$ by $B^2\mathbf{k}^\times$. There is a more analogous object in the setting of metaplectic quantization, rather than spin quantization. There is a well-known analogy between these two settings: instead of starting with a quadratic vector space, we start with a symplectic vector space. The Clifford algebra is replaced by the Weyl algebra, and the analogue of the Spin double-cover of SO is the double cover Mp of Sp. See [Del99] for a more in-depth explanation of the analogy.

The analogue of $\mathrm{Aut}_{\mathbf{EqBr}}(\mathcal{A})$ in the metaplectic setting is what is sometimes called the *affine symplectic group*:

$$A\,\mathrm{Sp}(V,\omega) = V \rtimes \mathrm{Sp}(V,\omega)\ .$$

Then the *extended symplectic group* $E\,\mathrm{Sp}$ is an extension of $A\,\mathrm{Sp}$ by $\mathrm{U}(1)$, and in fact

$$E\,\mathrm{Sp}(V,\omega) = \mathrm{Heis} \rtimes \mathrm{Mp}(V,\omega)\ ,$$

where Heis is the Heisenberg Lie group associated to (V,ω). I.e. we have a diagram:

$$\begin{array}{ccccc}
\mathrm{U}(1) & \longrightarrow & \mathrm{Heis} & \longrightarrow & V \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{U}(1) & \longrightarrow & E\,\mathrm{Sp}(V,\omega) & \longrightarrow & A\,\mathrm{Sp}(V,\omega) \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{Z}/2 & \longrightarrow & \mathrm{Mp} & \longrightarrow & \mathrm{Sp}
\end{array}$$

with exact rows and columns. This is analogous to the diagram:

$$\begin{array}{ccccc}
B^3\mathbf{k}^\times & \longrightarrow & B^2\mathcal{A}^\times & \longrightarrow & B^2A \\
\downarrow & & \downarrow & & \downarrow \\
B^3\mathbf{k}^\times & \longrightarrow & B\,\mathrm{Pic}(\mathcal{A}) & \longrightarrow & B\,\mathrm{Aut}_{\mathbf{EqBr}}(\mathcal{A}) \\
\downarrow & & \downarrow & & \downarrow \\
B^3\mu_{l^4} & \longrightarrow & B\,3\mathrm{Pin} & \longrightarrow & B\,\mathrm{O}(A,q)
\end{array}$$

There is an analogue of the Heisenberg Lie algebra in the Spin setting: It is the super Lie algebra generated by V with super-bracket determined by the quadratic form. The analogue of Heis should be some super group integrating this super Lie algebra.

Note that, even though $V \rtimes \mathrm{Sp}(V)$ acts on the Weyl algebra, it is not necessarily the whole automorphism group. I.e. see [BKK05] where the question of identifying the full automorphism group is discussed.

5. Projective 3-dimensional TQFTs

5.1. **Dualizability and invertibility.** It was shown in [DSPS20] that the category of fusion categories **Fus** has duals.[13] Therefore the Cobordism Hypothesis tells us that, given a fusion category $\mathcal{C}$, there is a uniquely defined fully-extended topological field theory

$$\begin{array}{ccc}
\mathbf{Bord}_3^{\mathrm{fr}} & \xrightarrow{F} & \mathbf{Fus} \\
* & \longmapsto & \mathcal{C}
\end{array} \tag{5.1}$$

This is sometimes called the *Turaev-Viro theory* associated to a fusion category [TV92, FT21].

[13] Technically our definition of **Fus** differs from the setting of finite semisimple abelian categories by an Ind-completion. However, as discussed in Remark 4.1, we can nonetheless apply the theorems of [DSPS20, ENO10] in our context.

It was shown in [BJS21] that the category of braided fusion categories **BrFus** has duals. Therefore, given a braided fusion category $\mathcal{A}$, the Cobordism Hypothesis provides a fully-extended topological field theory $\mathbf{Bord}_4^{\mathrm{fr}} \to \mathbf{BrFus}$ sending the point to $\mathcal{A}$. This is a framed version of the *Crane-Yetter theory* associated to $\mathcal{A}$ [CY93, FT21].

Remark 5.2. For ease of discussion, let $\mathcal{A}$ be the braided fusion category attached to a metric group (A, q). Consider the space of invertible objects in $\mathcal{A}$, written $\mathcal{A}^\times$. The fact that $\mathcal{A}$ was braided means we are able to deloop twice, to obtain a 3-type $B^2\mathcal{A}^\times$. The homotopy groups are all trivial except $\pi_2 = A$ and $\pi_3 = \mathbf{k}^\times$. The only remaining information needed to specify this space is the k-invariant in $H^4\left(B^2A, \mathbf{k}^\times\right)$. It is a theorem of Eilenberg-Mac Lane [EML54, Theorem 26.1] that there is an isomorphism $H^4\left(B^2A, \mathbf{k}^\times\right) \cong \mathrm{Quad}\left(A, \mathbf{k}^\times\right)$ (see (4.18)); write τ_q for the class corresponding to q.

Assuming Hypothesis Q, we can construct the theory associated to the π-finite space B^2A twisted by a cocycle representing τ_q, which turns out to be precisely the framed fully-extended Crane-Yetter theory from above sending the point to $\mathcal{A}$. This is discussed in Example B.9 (iv).

5.1.1. *Invertibility.* Invertibility of a braided tensor category was shown in [BJSS21] to be equivalent to checking three conditions. When the braided tensor category is fusion, these conditions become equivalent so we only have to check one. One of the conditions, *non-degeneracy*, asks if the *Müger center*[14] is trivial.

If $\mathcal{C}$ is a fusion category such that $\mathcal{A} \cong \mathcal{Z}(\mathcal{C})$, then the Müger center is known to be trivial [Müg03, DGNO10], so the TQFT sending the point to $\mathcal{Z}(\mathcal{C})$ is invertible; in fact it is trivial (see Section 5.3).

In particular, a nondegenerate braided fusion category $\mathcal{A}$ is invertible in **BrFus**, and therefore we have equivalences

$$\mathrm{Aut}_{\mathbf{BrFus}}(\mathcal{A}) \simeq \mathrm{Aut}_{\mathbf{BrFus}}(1) \simeq \mathbf{Fus}^\times \simeq B^3\mathbf{k}^\times \ . \tag{5.3}$$

5.2. **WRT theories.** Recall the description of the projective target category $\mathbb{P}\mathbf{Fus}$ from Section 2.3.3, and the corresponding notion of a projective TQFT from Definition 2.17. Let $\mathcal{A}$ be any nondegenerate braided fusion category. Note that the regular module $\mathcal{A}_\mathcal{A}$ defines a 1-morphism in **BrFus** from $\mathcal{A}$ to 1, and it is fully-dualizable as an object of the arrow category [Haï24]. Since $\mathcal{A}$ is fusion and nondegenerate, it is also an invertible object of **BrFus** [BJSS21]. In other words, the 1-morphism $\mathcal{A}_\mathcal{A} \colon \mathcal{A} \to 1$ is in the subcategory $\mathbb{P}\mathbf{Fus} \subset \mathbf{BrFus}^{\downarrow 1}$.

Summarizing this discussion, the cobordism hypothesis implies that $\mathcal{A}$ defines a fully-extended projective TQFT:

$$\mathbf{WRT}_\mathcal{A} \colon \mathbf{Bord}_3^{\mathrm{fr}} \to \mathbb{P}\mathbf{Fus} \ .$$

This is fully-extended anomalous Witten-Reshetikhin-Turaev (WRT)[15] theory, with a caveat: it is not oriented. This would be the data of a homotopy $\mathrm{SO}(3)$-fixed point structure on this object of $\mathbb{P}\mathbf{Fus}$.

The anomaly of $\mathbf{WRT}_\mathcal{A}$ sends the point to $\mathcal{A}$, so it is the restriction to $\mathbf{Bord}_3^{\mathrm{fr}}$ of the TQFT:

$$\mathbf{CY}_\mathcal{A} \colon \mathbf{Bord}_4^{\mathrm{fr}} \to \mathbf{BrFus}$$

which is classified by sending the point to $\mathcal{A}$. Recall this is a framed version of the *Crane-Yetter theory* associated to $\mathcal{A}$ [CY93, FT21].

As it turns out, $\mathbf{WRT}_\mathcal{A}$ has a canonical projective action of the 2-group of braided autoequivalences of $\mathcal{A}$.

---

[14] The Müger center of a braided tensor category consists of the objects which braid trivially with all other objects.

[15] Oftentimes WRT theory denotes the linear TQFT defined out of the extended bordism category as in Remark 3.10. As is explained there, these are equivalent.

Theorem 5.4. *Let $\mathcal{A}$ be a nondegenerate braided fusion category, and consider a monoidal functor $\Phi\colon G \to \mathrm{Aut}_{\mathbf{EqBr}}(\mathcal{A})$ for a group G. There is an anomalous TQFT*

$$\mathbf{WRT}^G_{\mathcal{A}}\colon \mathbf{Bord}_3^{BG} \to \mathbb{P}\mathbf{Fus}$$

which agrees with $\mathbf{WRT}_{\mathcal{A}}$ upon restriction to trivial G-bundles:

$$\begin{array}{ccc} \mathbf{Bord}^{BG} & & \\ \uparrow & \searrow^{\mathbf{WRT}^G_{\mathcal{A}}} & \\ \mathbf{Bord}^{fr} & \xrightarrow{\mathbf{WRT}_{\mathcal{A}}} & \mathbb{P}\mathbf{Fus} \end{array}$$

Furthermore, the anomaly theory of $\mathbf{WRT}^G_{\mathcal{A}}$ is the restriction to $\mathbf{Bord}_3^{BG}$ of a theory

$$\mathbf{CY}^G_{\mathcal{A}}\colon \mathbf{Bord}_4^{BG} \to \mathbf{BrFus}\ ,$$

which agrees with $\mathbf{CY}_{\mathcal{A}}$ upon restriction to trivial G-bundles.

Proof. By the Cobordism Hypothesis (A.6), symmetric monoidal functors from $\mathbf{Bord}_3^{BG}$ to $\mathbb{P}\mathbf{Fus}$ which restrict to $\mathbf{WRT}_{\mathcal{A}}$ are classified by maps:

$$BG \to B\,\mathrm{Aut}_{\mathbb{P}\mathbf{Fus}}\left(\mathcal{A} \xrightarrow{\mathcal{A}_{\mathcal{A}}} \mathbf{Vect}\right)\ .$$

There is a canonical functor:

$$\mathrm{Aut}_{\mathbf{EqBr}}(\mathcal{A}) \to \mathrm{Aut}_{\mathbf{BrFus}}(\mathcal{A})$$

given by sending a braided autoequivalence φ to the identity bimodule twisted by φ, written ${}_{\mathcal{A}}\mathcal{A}_{\varphi(\mathcal{A})}$. Precomposing with the given monoidal functor $\Phi\colon G \to \mathrm{Aut}_{\mathbf{EqBr}}(\mathcal{A})$ defines an action:

$$\overline{\Phi}\colon G \to \mathrm{Aut}_{\mathbf{BrFus}}(\mathcal{A})\ .$$

Now notice that Φ and $\overline{\Phi}$ assemble together into a functor

$$G \to \mathrm{Aut}_{\mathbb{P}\mathbf{Fus}}(\mathcal{A}_{\mathcal{A}}) \tag{5.5}$$

which on objects sends $g \in G$ to the lax square

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\overline{\Phi}(g)} & \mathcal{A} \\ \downarrow{\scriptstyle \mathcal{A}_{\mathcal{A}}} & \swarrow{\scriptstyle \Phi(g)} & \downarrow{\scriptstyle \mathcal{A}_{\mathcal{A}}} \\ \mathbf{Vect} & = & \mathbf{Vect} \end{array}$$

Here the 2-morphism filling the square is the equivalence of $\mathcal{A}$-modules $\mathcal{A}_{\mathcal{A}} \otimes_{\mathcal{A}} {}_{\mathcal{A}}\mathcal{A}_{\Phi(g)(\mathcal{A})} \simeq \mathcal{A}_{\mathcal{A}}$ given by the braided autoequivalence $\Phi(g)$ itself. Since this 2-cell is invertible, its orientation does not matter.

This assignment is compatible with composition in G, and with all higher coherences, because it is the restriction of a map from the space of nondegenerate braided fusion categories and braided equivalences to the space of objects of $\mathbb{P}\mathbf{Fus}$, sending $\mathcal{B}$ to its regular module $\mathcal{B}_{\mathcal{B}}$ and a braided equivalence $\varphi\colon \mathcal{B} \to \mathcal{B}'$ to the square above with top arrow ${}_{\mathcal{B}}\mathcal{B}'_{\varphi}$. The regular module and the twisted bimodules are both induced central algebras [Haï24, Definition 3.8], and this construction preserves composition [Haï24, Proposition 3.9]. Restricting to the path component of $\mathcal{A}$ gives a map $B\,\mathrm{Aut}_{\mathbf{EqBr}}(\mathcal{A}) \to B\,\mathrm{Aut}_{\mathbb{P}\mathbf{Fus}}(\mathcal{A}_{\mathcal{A}})$, and we compose it with $B\Phi\colon BG \to B\,\mathrm{Aut}_{\mathbf{EqBr}}(\mathcal{A})$. Note that this classifies a functor

$$\mathbf{WRT}^G_{\mathcal{A}}\colon \mathbf{Bord}_3^{BG} \to \mathbb{P}\mathbf{Fus}\ ,$$

and its restriction to trivial bundles sends the point to the regular module, and therefore agrees with $\mathbf{WRT}_{\mathcal{A}}$ by the cobordism hypothesis. The anomaly theory of $\mathbf{WRT}^G_{\mathcal{A}}$ is its composition with

the functor $s\colon \mathbb{P}\mathbf{Fus} \to \mathbf{BrFus}^{\times}$ from Definition 2.17. It is the restriction to $\mathbf{Bord}_3^{BG}$ of the theory $\mathbf{CY}_{\mathcal{A}}^G\colon \mathbf{Bord}_4^{BG} \to \mathbf{BrFus}$ classified, by the Cobordism Hypothesis (A.6), by the composite

$$BG \to B\operatorname{Aut}_{\mathbb{P}\mathbf{Fus}}(\mathcal{A}_{\mathcal{A}}) \to B\operatorname{Aut}_{\mathbf{BrFus}}(\mathcal{A}) \simeq B^4\mathbf{k}^{\times}$$

of the map above with the map induced by s (see (5.3)), i.e. by a 4-cocycle on G. On trivial G-bundles this classifying map is constant at $\mathcal{A}$, so $\mathbf{CY}_{\mathcal{A}}^G$ agrees with $\mathbf{CY}_{\mathcal{A}}$ there. □

Remark 5.6. Recall from Remark 3.10 that, given an anomalous TQFT, there is an extended bordism category over which the anomaly can always be trivialized. As is explained in [Wal06, Fre12, BDSPV15, Haï25], if you lift from the oriented 4-dimensional bordism category to the bordism category of oriented 4-manifolds equipped with a trivialization of p_1, then the Crane-Yetter theory can be trivialized. Another option is equipping the 4-manifolds with a signature structure in the sense of [BDSPV15]. Choices of the latter structure correspond to square roots of the central charge, and the choices of the former correspond to sixth roots, so these relate to one another by a factor of three [BDSPV15].

Remark 5.7. If the anomaly theory happens to be trivializable as a functor out of $\mathbf{Bord}^{BG}$, then the trivializations $1 \xrightarrow{\sim} \mathbf{CY}^G$ will form a torsor over invertible theories of one dimension lower. We will save a more detailed discussion of trivializations for the more restricted examples of 3-dimensional theories in the coming sections.

5.3. **The anomaly and the center.** Consider the following theory ζ.

$$\begin{array}{ccc} \mathbf{Bord}_4^{\mathrm{fr}} & \xrightarrow{\zeta} & \mathbf{BrFus} \\ * & \longmapsto & \mathcal{Z}(\mathcal{C}) \end{array}$$

Note that $\mathcal{C}$ always defines a module ${}_{\mathcal{Z}}\mathcal{C}$ over its own center $\mathcal{Z}(\mathcal{C})$. As for $\mathcal{A}_{\mathcal{A}}$ in Section 5.2, this is a 1-morphism ${}_{\mathcal{Z}}\mathcal{C}\colon \mathcal{Z}(\mathcal{C}) \to 1$ in **BrFus**, and by [JFS17, Theorem 7.15] it classifies a relative theory, i.e. a lax natural transformation (Remark A.11):

$$F_{\zeta}\colon \zeta \to 1\ .$$

By Section 5.1.1, ζ is invertible, so we can think of F_{ζ} as an anomalous theory, with anomaly the restriction of ζ to $\mathbf{Bord}_3^{\mathrm{fr}}$.

In fact, the Drinfeld center of a fusion category $\mathcal{C}$ is trivial in **BrFus**: $\mathcal{C}$ defines an equivalence between 1 and $\mathcal{Z}(\mathcal{C})$. See the proof of [BJSS21, Theorem 4.2]. For a general braided fusion category $\mathcal{A}$, an $\mathcal{A}$-central fusion category $\mathcal{C}$ was shown to give an equivalence between $\mathcal{A}$ and **Vect** if and only if the natural map $\mathcal{A} \to \mathcal{Z}(\mathcal{C})$ is an equivalence in [JMPP22, Theorem 2.23].

Remark 5.8. Recall $\operatorname{Pic}(\mathcal{Z}(\mathcal{C}))$ from Section 4.2. Note that these modules are not 1-morphisms in **BrFus** (they are only modules, not central modules). They are however 2-morphisms in **BrFus**. In the language of defects, this is saying that Pic does not consist of domain walls, but rather certain codimension-2 defects. In particular, consider a domain wall between the trivial theory and ζ labelled by $\mathcal{Z}(\mathcal{C})$ as a module over itself. Now the objects of Pic label self-interfaces between this domain wall.

5.4. **Projective action of the braided automorphisms of the center.** Fix a TQFT F associated to $\mathcal{C} \in \mathbf{Fus}$ as in (5.1). We will apply Corollary 3.17 to this theory.

Recall the notation from Section 4.4.4. In particular, recall the class from (4.39):

$$k_3\colon B\operatorname{Aut}_{\mathbf{EqBr}}(\mathcal{Z}(\mathcal{C})) \to B^4\mathbf{k}^{\times}\ .$$

In the notation of Section 3.2, we will let $X = B\operatorname{Aut}_{\mathbf{EqBr}}(\mathcal{Z}(\mathcal{C}))$ and $c = k_3$, so

$$\widetilde{X} = B\operatorname{Aut}_{\mathbf{Fus}}(\mathcal{C})\ .$$

Since $\pi_3 B\operatorname{Aut}_{\mathbf{Fus}}(\mathcal{C}) \cong \mathbf{k}^\times$ (see Section 4.4.3), $\mathcal{C}$ is a Schur object (Definition 3.14), and therefore Corollary 3.17 applies.

By the general discussion of Section 3.2, there is an anomaly theory, i.e. a once categorified 3-dimensional TQFT

$$\alpha_{k_3} \colon \mathbf{Bord}_3^{B\operatorname{Aut}_{\mathbf{EqBr}}(\mathcal{Z}(\mathcal{C}))} \to \mathbf{BrFus} \ ,$$

classified by k_3.

Remark 5.9. This is a situation where the anomaly theory α_{k_3} actually extends to a $(d+1) = 4$-dimensional theory, i.e. a functor out of $\mathbf{Bord}_4^{B\operatorname{Aut}_{\mathbf{EqBr}}(\mathcal{Z}(\mathcal{C}))}$, by construction (Section 3.2).

Theorem 5.10. *The framed TQFT F has an $X = B\operatorname{Aut}_{\mathbf{EqBr}}(\mathcal{Z}(\mathcal{C}))$-anomaly as in Definition 3.4. In particular, the fusion category $\mathcal{C}$ itself defines an anomalous theory:*

$$F_{k_3} \colon \alpha_{k_3} \to 1_{B\operatorname{Aut}_{\mathbf{EqBr}}(\mathcal{Z}(\mathcal{C}))} \ . \tag{5.11}$$

Furthermore, if $\mathcal{Z}(\mathcal{C})$ is pointed, and the cohomology class classifying the braiding of $\mathcal{Z}(\mathcal{C})$ (as in Remark 5.2) is nontrivial (e.g. whenever $\mathcal{C} \ncong \mathbf{Vect}$), then the class k_3 classifying α_{k_3} is nontrivial.

Proof. As remarked above, $\mathcal{C}$ is a Schur object (Definition 3.14) of **Fus**, so Corollary 3.17 applies. The fact that $F \colon * \mapsto \mathcal{C}$ has an X-anomaly is then Corollary 3.17(ii).

Recall from Remark 5.2 that the class classifying the braiding is the same as the k-invariant of $B^2\mathcal{Z}^\times$. Also recall the Postnikov and Whitehead towers of $B\operatorname{Aut}_{\mathbf{Fus}}(\mathcal{C})$ from (4.31). The composition of k_3 with the map from the universal cover agrees with the k-invariant of $B^2\mathcal{Z}^\times$ over $B^2\pi_2$. Indeed, k_3 classifies the fibration $B^3\mathbf{k}^\times \to B\operatorname{Aut}_{\mathbf{Fus}}(\mathcal{C}) \to B\operatorname{Aut}_{\mathbf{EqBr}}(\mathcal{Z}(\mathcal{C}))$, and pulling this fibration back along the universal cover $B^2\pi_2 \to B\operatorname{Aut}_{\mathbf{EqBr}}(\mathcal{Z}(\mathcal{C}))$ gives the 1-connected cover $B^2\mathcal{Z}(\mathcal{C})^\times$ of $B\operatorname{Aut}_{\mathbf{Fus}}(\mathcal{C})$ (see (4.31)), so the pulled back class is the k-invariant of $B^2\mathcal{Z}(\mathcal{C})^\times$. I.e. the following diagram commutes:

$$\begin{array}{ccc}
B^2\mathcal{Z}(\mathcal{C})^\times & \longrightarrow & B\operatorname{Aut}_{\mathbf{Fus}}(\mathcal{C}) \\
\downarrow & & \downarrow \\
B^2\pi_2 & \longrightarrow & B\operatorname{Aut}_{\mathbf{EqBr}}(\mathcal{Z}(\mathcal{C})) \\
& \searrow^{\tau_q} & \searrow^{k_3} \\
& & B^4\mathbf{k}^\times
\end{array}$$

where we write τ_q for the class corresponding to q as in Remark 5.2. From this diagram we see that trivial k_3 implies trivial τ_q. Therefore if τ_q is nontrivial, then k_3 is nontrivial. If $\mathcal{C} \ncong \mathbf{Vect}$ then $\operatorname{FPdim}\mathcal{Z}(\mathcal{C}) = \operatorname{FPdim}(\mathcal{C})^2 > 1$ [EGNO15], so $A \neq 0$, and a nondegenerate quadratic form on a nonzero group is nontrivial; hence τ_q is nontrivial. □

5.4.1. *Trivializing the anomaly.* Recall the setting of Section 4.4.4. If we can trivialize k_3 over

$$f_2 \colon BG_2 \to B\operatorname{Aut}_{\mathbf{EqBr}}(\mathcal{Z}(\mathcal{C})) \ ,$$

then the trivializations form a torsor over $H^3(BG_2, \mathbf{k}^\times)$, and a single trivialization determines

$$1_{BG_2} \xrightarrow{\sim} \alpha_{f_2^* k_3} \to 1_{BG_2} \ ,$$

which is a (non-anomalous) G_2-theory by Proposition A.9. This is Corollary 3.17(b). The equivalence with Corollary 3.17(c) and (d) in this example will be discussed in Sections 5.6 and 5.8.

5.5. **Projective action of the orthogonal group.** Still fix $F\colon * \mapsto \mathcal{C}$ (5.1), as in Sections 5.1, 5.3, and 5.4. Although now we will assume that we are in the setting of Section 4.4.3. I.e. we are assuming that the obstruction $O_3(f)$ from (4.32), associated to a map

$$f\colon G \to \pi_1 B\operatorname{Aut}_{\mathbf{Fus}}(\mathcal{C}) \ ,$$

is trivializable, and that we have chosen a trivialization s. Recall the class

$$O_4(f,s) \in H^4\left(BG, \mathbf{k}^\times\right)$$

from (4.37).

We will apply Theorem 3.12 to this setting. I.e. let $\mathcal{T} = \mathbf{BrFus}$, as in Section 5.4. Set $X = BG$ and $c = O_4(f,s)$. By the general discussion of Section 3.2, there is an anomaly theory

$$\alpha_{O_4(f,s)}\colon \mathbf{Bord}_3^{BG} \to \mathbf{BrFus} \tag{5.12}$$

classified by $O_4(f,s)$.

By definition, the space classified by $O_4(f,s)$ is the pullback:

$$\begin{array}{ccc} \operatorname{hofib}(O_4(f,s)) & \longrightarrow & B\operatorname{Aut}_{\mathbf{Fus}}(\mathcal{C}) \\ \downarrow & & \downarrow \\ BG & \xrightarrow{s} & \pi_{\leq 2} B\operatorname{Aut}_{\mathbf{Fus}}(\mathcal{C}) \end{array}$$

The inclusion of the connected component corresponding to $\mathcal{C}$ defines a map:

$$B\operatorname{Aut}_{\mathbf{Fus}}(\mathcal{C}) \to \mathbf{Fus} \ ,$$

so composition with the map defined by the pullback gives us:

$$\operatorname{hofib}(O_4(f,s)) \to B\operatorname{Aut}_{\mathbf{Fus}}(\mathcal{C}) \to \mathbf{Fus} \ ,$$

which classifies a TQFT (by the Cobordism Hypothesis (A.6)):

$$\mathbf{Bord}_3^{\operatorname{hofib}(O_4(f,s))} \to \mathbf{Fus} \ .$$

This theory is the extended TQFT associated to the anomalous theory

$$F_{O_4(f,s)}\colon \alpha_{O_4(f,s)} \to 1_{BG}$$

by Proposition 3.8.

5.5.1. *Pointed center.* Continuing the investigation of TQFTs of the form (5.1), we now assume that $\mathcal{Z}(\mathcal{C})$ is pointed. As in Proposition 4.29, this is classified by its group A of simple objects and a quadratic form $q\colon A \to \mathbf{k}^\times$ which describes the braiding.

We assume $|A|$ is odd, so that by Section 4.4.5 there is a canonical splitting s of $B\operatorname{Aut}_{\mathbf{EqBr}}(\mathcal{Z}(\mathcal{C}))$. Therefore there is a universal class defined in (4.40):

$$O_4 = O_4(A,q) \in H^4\left(B\,\mathrm{O}(A,q), \mathbf{k}^\times\right) \ .$$

By the general discussion in Section 3.2, there is an anomaly classified by O_4:

$$\alpha_{O_4}\colon \mathbf{Bord}_3^{B\,\mathrm{O}(A,q)} \to \mathbf{BrFus} \ . \tag{5.13}$$

Recall the 3-group $3\mathrm{Lip}(A,q)$ defined in Section 4.5. By definition, the space classified by O_4 is the pullback:

$$\begin{array}{ccc} B\,3\mathrm{Lip}(A,q) = \operatorname{hofib}(O_4) & \longrightarrow & B\operatorname{Aut}_{\mathbf{Fus}}(\mathcal{C}) \\ \downarrow & & \downarrow \\ B\,\mathrm{O}(A,q) & \xrightarrow{s} & \pi_{\leq 2} B\operatorname{Aut}_{\mathbf{Fus}}(\mathcal{C}) \end{array}$$

where s is the canonical splitting

$$s\colon B\,\mathrm{O}(A,q) \to B\operatorname{Aut}_{\mathbf{EqBr}}(\mathcal{Z}(\mathcal{C})) \simeq \pi_{\leq 2} B\operatorname{Aut}_{\mathbf{Fus}}(\mathcal{C}) \ .$$

The inclusion of the connected component corresponding to $\mathcal{C}$ defines a map:

$$B\operatorname{Aut}_{\mathbf{Fus}}(\mathcal{C}) \to \mathbf{Fus}\ ,$$

so composition gives us a map:

$$B\,3\mathrm{Lip}(A,q) \to B\operatorname{Aut}_{\mathbf{Fus}}(\mathcal{C}) \to \mathbf{Fus}\ , \tag{5.14}$$

which classifies a TQFT:

$$F_{3\mathrm{Lip}}\colon \mathbf{Bord}_3^{B\,3\mathrm{Lip}(A,q)} \to \mathbf{Fus}$$

by the Cobordism Hypothesis (A.6). This is the extended theory corresponding to the anomalous theory

$$F_{O_4}\colon \alpha_{O_4} \to 1_{B\,\mathrm{O}(A,q)}$$

by Proposition 3.8.

The same holds when we restrict to the 3-groups $3\mathrm{Pin}(A,q)$ and $3\mathrm{Spin}(A,q)$ defined in Section 4.5 as follows. In particular, recall the cohomology classes (4.49) and (4.51):

$$c(3\mathrm{Pin}) \in H^4(B\,\mathrm{O}(A,q),\mu_{l^4}) \qquad c(3\mathrm{Spin}) \in H^4(B\,\mathrm{SO}(A,q),\mu_{l^4})\ .$$

These classify anomaly theories, by the general discussion in Section 3.2:

$$\alpha_{c(3\mathrm{Pin})}\colon \mathbf{Bord}_3^{B\,\mathrm{O}(A,q)} \to \mathbf{BrFus} \tag{5.15}$$

$$\alpha_{c(3\mathrm{Spin})}\colon \mathbf{Bord}_3^{B\,\mathrm{SO}(A,q)} \to \mathbf{BrFus}\ . \tag{5.16}$$

We have a map, e.g. from (4.50), $3\mathrm{Pin}(A,q) \to 3\mathrm{Lip}(A,q)$. We also have a map $3\mathrm{Spin}(A,q) \to 3\mathrm{Pin}(A,q)$ from Proposition 4.53, which we can compose with the map $3\mathrm{Pin}(A,q) \to 3\mathrm{Lip}(A,q)$ to obtain a map $3\mathrm{Spin}(A,q) \to 3\mathrm{Lip}(A,q)$. The upshot of this, is that we can compose these maps with (5.14) to obtain functors:

$$B\,3\mathrm{Pin}(A,q) \to \mathbf{Fus} \qquad B\,3\mathrm{Spin}(A,q) \to \mathbf{Fus}$$

which classify a $3\mathrm{Pin}(A,q)$-TQFT and a $3\mathrm{Spin}(A,q)$-TQFT:

$$F_{3\mathrm{Pin}}\colon \mathbf{Bord}_3^{B\,3\mathrm{Pin}(A,q)} \to \mathbf{Fus}$$

$$F_{3\mathrm{Spin}}\colon \mathbf{Bord}_3^{B\,3\mathrm{Spin}(A,q)} \to \mathbf{Fus}$$

by the Cobordism Hypothesis (A.6). These are the pullbacks, along $B\,3\mathrm{Pin}(A,q) \to B\,3\mathrm{Lip}(A,q)$ (resp. $B\,3\mathrm{Spin}(A,q) \to B\,3\mathrm{Lip}(A,q)$), of the extended theory $F_{3\mathrm{Lip}}$, which by Proposition 3.8 is associated to the anomalous theories:

$$F_{c(3\mathrm{Pin})}\colon \alpha_{c(3\mathrm{Pin})} \to 1_{B\,\mathrm{O}(A,q)}$$

$$F_{c(3\mathrm{Spin})}\colon \alpha_{c(3\mathrm{Spin})} \to 1_{B\,\mathrm{SO}(A,q)}$$

(recall from Remark 5.19 that $\alpha_{c(3\mathrm{Pin})} \simeq \alpha_{O_4}$), where $1_{B\,\mathrm{O}(A,q)}$ (resp. $1_{B\,\mathrm{SO}(A,q)}$) denotes the trivial **BrFus**-valued once-categorified theory defined on $\mathbf{Bord}_3^{B\,\mathrm{O}(A,q)}$ (resp. $\mathbf{Bord}_3^{B\,\mathrm{SO}(A,q)}$).

Summarizing this discussion, we have the following.

Theorem 5.17. *Let $\mathcal{C}$ be a fusion category with pointed Drinfeld center. Recall from Proposition 4.29 that $\mathcal{Z}(\mathcal{C})$ is classified by a metric group (A,q) which has a Lagrangian subgroup, and assume $|A|$ is odd. The nonanomalous framed theory*

$$F\colon \mathbf{Bord}_3^{fr} \to \mathbf{Fus}$$

sending the point to $\mathcal{C}$ has an $\mathrm{O}(A,q)$-anomaly in the sense of Definition 3.4. I.e. there is an anomaly theory

$$\alpha_{c(3\mathrm{Pin})}\colon \mathbf{Bord}_3^{B\,\mathrm{O}(A,q)} \to \mathbf{BrFus}\ ,$$

and $\mathcal{C}$ canonically defines an anomalous $\mathrm{O}(A,q)$-TQFT:

$$F_{c(3\mathrm{Pin})}\colon \alpha_{c(3\mathrm{Pin})} \to 1_{B\,\mathrm{O}(A,q)}\ . \tag{5.18}$$

Remark 5.19. By construction (Section 3.2), the anomaly theory $\alpha_{c(3\mathrm{Pin})}$ only depends on the image of $c\,(3\mathrm{Pin})$ in $H^4\left(B\,\mathrm{O}\,(A,q)\,,\mathbf{k}^\times\right)$, which is $O_4\,(A,q)$ for every choice of the lift $c\,(3\mathrm{Pin})$ in Section 4.5. Hence $\alpha_{c(3\mathrm{Pin})} \simeq \alpha_{O_4}$, and Theorem 5.17 and Corollaries 5.21, 5.27 and 5.33 do not depend on this choice, although the 3-groups $3\mathrm{Pin}\,(A,q)$ and $3\mathrm{Spin}\,(A,q)$ do.

Remark 5.20. As in Section 5.4, the anomalies $\alpha_{O_4(f,s)}$, and therefore α_{O_4}, $\alpha_{c(3\mathrm{Pin}(A,q))}$, and $\alpha_{c(3\mathrm{Spin})}$, defined in (5.12), (5.13), (5.15), and (5.16) extend to $(d+1)=4$-dimensional theories, i.e. functors out of $\mathbf{Bord}_4^{BG}$ (for $G=\mathrm{O}\,(A,q)$ or $\mathrm{SO}\,(A,q)$) by construction (Section 3.2).

Corollary 5.21. *Restricting the* $\mathrm{O}\,(A,q)$*-anomaly of Theorem 5.17 to* $\mathrm{SO}\,(A,q)$ *we obtain an* $\mathrm{SO}\,(A,q)$*-anomaly of* F

$$\alpha_{c(3\mathrm{Spin})}\colon \mathbf{Bord}_3^{B\,\mathrm{SO}(A,q)} \to \mathbf{BrFus}\ ,$$

and an anomalous $\mathrm{SO}\,(A,q)$*-equivariant theory*

$$F_{c(3\mathrm{Spin})}\colon \alpha_{c(3\mathrm{Spin})} \to 1_{B\,\mathrm{SO}(A,q)}\ . \tag{5.22}$$

5.5.2. *Trivializing the anomaly.* As in Section 3.2.1, we ask if we can trivialize the anomaly from Theorem 5.17 (resp. Corollary 5.21) after pulling back along a map $f\colon G \to \mathrm{O}\,(A,q)$ (resp. $f\colon G\to \mathrm{SO}\,(A,q)$) for G a finite group.

The map f defines a new anomaly theory $\alpha_{f^*c(3\mathrm{Pin})}$:

$$\begin{array}{ccc} \mathbf{Bord}_3^{BG} & & \\ \downarrow{\scriptstyle f} & \searrow^{\alpha_{f^*c(3\mathrm{Pin})}} & \\ \mathbf{Bord}_3^{B\,\mathrm{O}(A,q)} & \xrightarrow[\alpha_{c(3\mathrm{Pin})}]{} & \mathbf{BrFus}\ . \end{array}$$

By Remark 5.19, $\alpha_{f^*c(3\mathrm{Pin})} \simeq \alpha_{f^*O_4}$, so a trivialization of $f^*O_4\,(A,q)$ in $H^4\,(BG,\mathbf{k}^\times)$ trivializes it. A trivialization of $f^*c\,(3\mathrm{Pin}) \in H^4\,(BG,\mu_{l^4})$ induces one, but not conversely in general. A trivialization of the anomaly $\alpha_{f^*c(3\mathrm{Pin})}$ yields a (non-anomalous) G-theory by Proposition A.9

$$1_{BG} \to \alpha_{f^*c(3\mathrm{Pin})} \to 1_{BG}\ .$$

In addition to a fusion category $\mathcal{C}$, the data defining a theory $\mathbf{Bord}_3^{BG} \to \mathbf{Fus}$ includes a coherent action of G on $\mathcal{C}$. This means each element $g\in G$ is assigned a bimodule M_g, along with equivalences between the composition $\mathcal{M}_h\otimes\mathcal{M}_g$ and $\mathcal{M}_{gh}$ (giving rise to vanishing O_3 as in Section 4.4.2) and associators giving rise to vanishing O_4. (See [ENO10, §7,8] for details.) This makes

$$\bigoplus_{g\in G} M_g \tag{5.23}$$

itself a fusion category, which is called a G-extension of $\mathcal{C}$.

Remark 5.24. [ENO10, Theorem 1.3] associates a G-extension of $\mathcal{C}$ to a trivialization of O_3 and O_4. If we use this trivialization to trivialize α, the G-extension of [ENO10] is precisely (5.23) from the G-theory $1_{BG}\to\alpha_{O_4(f,s)}\to 1_{BG}$.

5.6. **Projective 3d TQFTs.** In order to describe the projectivity captured by the anomalies in Theorems 5.10 and 5.17 and Corollary 5.21 in terms of projective theories, as in Theorem 3.12 and Corollary 3.17, we consider the projectivization (Definition 2.3) of **Fus** discussed in Section 2.3.3.

Definition 5.25. Projective 3-dimensional TQFTs with tangential structure (X,ζ) are symmetric monoidal functors

$$\mathbf{Bord}_3^{(X,\zeta)} \to \mathbb{P}\mathbf{Tens}\ ,$$

where $\mathbb{P}\mathbf{Tens}$ denotes the projectivization of $\Omega\mathbf{BrTens} \simeq \mathbf{Tens}$. We will primarily restrict our attention to theories factoring through the subcategory $\mathbb{P}\mathbf{Fus}$. The *projectivity* of $\overline{F}$ is the theory α (as in Definition 2.17):

$$\mathbf{Bord}_3 \xrightarrow{\overline{F}} \mathbb{P}\mathbf{Fus} \xrightarrow{s} \mathbf{BrFus}^{\times} \ .$$
$$\mathbf{Bord}_3 \xrightarrow{\alpha} \mathbf{BrFus}^{\times} \ .$$

Let $\mathcal{C}$ be an object of the Morita 3-category of fusion categories. As in (5.1), this classifies a framed theory $F\colon * \mapsto \mathcal{C}$.

Corollary 5.26. *As in Theorem 5.10, write $X = B\operatorname{Aut}_{\mathbf{EqBr}}(\mathcal{Z}(\mathcal{C}))$, and let k_3 be the cocycle in (4.35).*

The anomalous theory F_{k_3} in (5.11) is equivalent to some projective theory:

$$\overline{F}\colon \mathbf{Bord}_3^X \to \mathbb{P}\mathbf{Fus}$$

with underlying theory F, in the sense of Definition 2.19.

Corollary 5.26 follows from Corollary 3.17(iii) in the context of Theorem 5.10.

Let $c(\mathrm{3Pin})$ and $c(\mathrm{3Spin})$ be the cocycles defined in (4.49) and (4.51), which classify the anomaly theories $\alpha_{c(\mathrm{3Pin})}$ and $\alpha_{c(\mathrm{3Spin})}$ from (5.15) and (5.16).

Corollary 5.27. *Let $\mathcal{Z}(\mathcal{C})$ be pointed, classified by a metric group (A, q) as in Proposition 4.29, and assume $|A|$ is odd. The anomalous theory $F_{c(\mathrm{3Pin})}$ from (5.18) is equivalent to some projective theory:*

$$\overline{F}\colon \mathbf{Bord}_3^{B\,\mathrm{O}(A,q)} \to \mathbb{P}\mathbf{Fus}$$

with underlying theory F, in the sense of Definition 2.19.

The same result holds when $F_{c(\mathrm{3Pin})}$ is replaced with $F_{c(\mathrm{3Spin})}$ from (5.22), and $\mathrm{O}(A, q)$ is replaced with $\mathrm{SO}(A, q)$.

Corollary 5.27 follows from Theorem 3.12(i) in the context of Theorem 5.17 and Corollary 5.21.

As in the general discussion in Section 2.4 (namely Proposition 2.18 and Definition 2.17) a trivialization of any of the anomaly theories α_{k_3}, $\alpha_{c(\mathrm{3Pin})}$, or $\alpha_{c(\mathrm{3Spin})}$ determines a linearization of $\overline{F}$, which tautologically agrees with the trivialized anomalous theory $1 \xrightarrow{\sim} \alpha \xrightarrow{F_c} 1$. This is Theorem 3.13(i) (or Corollary 3.17(c)) in this context.

On the other hand, nontrivial cocycles on finite groups *always* describe anomalies of some 3-dimensional TQFT by the following.

Corollary 5.28. *For any nontrivial $\pi\colon BG \to B^4\mathbf{k}^{\times}$, there exists a π-finite space X and an anomaly TQFT*

$$\alpha_\pi\colon \mathbf{Bord}_3^X \to \mathbf{BrFus}$$

such that there exists a map $f\colon BG \to X$ such that the pullback of the cocycle classifying α along f agrees with the cohomology class of π.

Proof. Proposition 4.45 provides us with a nondegenerate braided fusion category $\mathcal{B}$ and $\Phi\colon G \to \operatorname{Aut}_{\mathbf{EqBr}}(\mathcal{B})$ such that $(B\Phi)^* k_3 = [\pi]$. Setting $X = B\operatorname{Aut}_{\mathbf{EqBr}}(\mathcal{B})$, $f = B\Phi$, and $\alpha_\pi = \alpha_{k_3}$ the anomaly theory classified by k_3 (as in Section 3.2), the result follows from Proposition 4.45. □

5.7. **Functorial assignment of TQFTs to a vector space over a finite field.** Let $L \simeq \mathbb{F}_p^n$ be a vector space over a finite field $\mathbb{F}_p$ ($p \neq 2$). The fusion category $\mathcal{C} = (\mathbf{Vect}[L], *)$ defines a theory:

$$F\colon \mathbf{Bord}_3^{\mathrm{fr}} \to \mathbf{Fus} \ .$$

Note that

$$\mathcal{Z}(\mathcal{C}) \simeq \mathbf{Vect}\left[\mathbb{F}_p^{2n}\right]$$

with braiding induced by the quadratic form q_{split} on $V \simeq \mathbb{F}_p^{2n}$ of signature (n,n). Write $\mathrm{O}(n,n;\mathbb{F}_p)$ for the split orthogonal group over $\mathbb{F}_p$. Recall this agrees with $\mathrm{O}(V, q_{\text{split}})$ (Observation 4.44).

Recall from Section 4.4.6 that the obstruction $O_4\left(\mathbb{F}_p^{2n}, q_{\text{split}}\right)$ is shown to vanish in this case in [EG18, Theorem 6.1]. Therefore we have the following by Theorem 5.17.

Corollary 5.29. *There is a canonically defined $H^3(B\,\mathrm{O}(n,n;\mathbb{F}_p), \mathbf{k}^\times)$-torsor of $\mathrm{O}(n,n;\mathbb{F}_p)$-TQFTs*

$$F_{O_{n,n}} : \mathbf{Bord}_3^{B\,\mathrm{O}(n,n;\mathbb{F}_p)} \to \mathbf{Fus}$$

with underlying framed theory F.

Recall the analogy between this categorical representation of $\mathrm{O}(n,n;\mathbb{F}_p)$ and the Weil representation of the metaplectic group in Remark 4.58. Corollary 5.29 can be thought as an analogue of the fact that the Weil representation splits over a finite field [GH09, GH11].

5.8. **Module structures on 3d TQFTs.** To describe the projectivity captured by the anomalies in Theorems 5.10 and 5.17 and Corollary 5.21 in terms of twisted module structures as in Theorem 3.12 and Corollary 3.17, we need to assume Hypothesis Q for our 4-dimensional target $\mathcal{T}$.

There is latitude in the choice of target. One option is to use $\mathcal{T} = \mathbf{BrFus}$. This target is sufficient to support our anomaly theories α_{k_3}, $\alpha_{c(3\text{Pin})}$, and $\alpha_{c(3\text{Spin})}$ from Theorems 5.10 and 5.17 and Corollary 5.21.

Definition 5.30. We say a 4-dimensional target $\mathcal{T}$ is **BrFus**-*compatible* if:

(a) $\Omega\mathcal{T} = \mathrm{End}_{\mathcal{T}}(1) \simeq \mathbf{Fus}$, and
(b) there is a fully-faithful symmetric monoidal functor $\mathbf{BrFus} \to \mathcal{T}$.

In this case we can regard the anomaly theories α_{k_3}, $\alpha_{c(3\text{Pin})}$, and $\alpha_{c(3\text{Spin})}$ from Theorems 5.10 and 5.17 and Corollary 5.21 as valued in $\mathcal{T}$.

Example 5.31. As is remarked in [DR18], one can define an $(\infty,4)$-category of algebras in the 3-category of finite semisimple 2-categories (with bimodules, linear 2-functors, natural transformations, and modifications). It is conjectured in [DR18] that 2-fusion categories are the fully-dualizable objects of this $(\infty,4)$-category.

Finite nonabelian topological 4-dimensional G-gauge theory is simply classified by sending the point to the fusion 2-category $2\mathbf{Vect}[G]$ of G-graded 2-vector spaces [DR18, Construction 2.1.13].

Let $\mathcal{C}$ be an object of the Morita 3-category of fusion categories. As in (5.1), this classifies a framed theory $F\colon * \mapsto \mathcal{C}$.

Corollary 5.32. *Assuming Hypothesis Q for a* **BrFus**-*compatible target $\mathcal{T}$ (Definition 5.30), the framed theory F has a canonical $\left(\sigma^4_{B\,\mathrm{Aut}_{\mathbf{EqBr}}(\mathcal{Z}(\mathcal{C})),k_3}, \rho\right)$-module structure.*

Proof. On account of the anomalous theory (5.11), Corollary 3.17 (iv) implies the result. □

Let G_2 be a finite 2-group (i.e. BG_2 is a π-finite 2-type), let $f\colon BG_2 \to X$ be a map, and assume we have chosen a trivialization of f^*k_3. Assuming Hypothesis Q, by Corollary 3.17(d), we can reduce (in the sense of Definition B.24) the canonical $\left(\sigma^4_{B\,\mathrm{Aut}_{\mathbf{EqBr}}(\mathcal{Z}(\mathcal{C})),k_3}, \rho\right)$-module structure on F in Corollary 5.32 to a $(\sigma_{BG_2}, \rho_{BG_2})$-module structure. This in particular yields a G_2-theory by Proposition B.14, which agrees with the G_2-theory obtained by trivializing the anomaly.

Corollary 5.33. *Let $\mathcal{Z}(\mathcal{C})$ be pointed, classified by a metric group (A,q) with $|A|$ odd. Assuming Hypothesis Q for a* **BrFus**-*compatible target $\mathcal{T}$ (Definition 5.30), there is a $\left(\sigma_{B\,\mathrm{O}(A,q),c(3\text{Pin})}, \rho_{B\,\mathrm{O}(A,q),c(3\text{Pin})}\right)$-module structure on F. Similarly, there is a $\left(\sigma_{B\,\mathrm{SO}(A,q),c(3\text{Spin})}, \rho_{B\,\mathrm{SO}(A,q),c(3\text{Spin})}\right)$-module structure on F.*

Proof. On account of the anomalous theory (5.18), Theorem 3.12(iii) implies the result. The $3\mathrm{Spin}(A,q)$ version follows from Theorem 3.12(iii), for the anomalous theory (5.22). □

Let $f\colon G \to \mathrm{O}(A,q)$ for a finite group G. Given a trivialization of $(Bf)^* O_4(A,q)$ (for instance one induced by a trivialization of $(Bf)^* c(3\mathrm{Pin}) \in H^4(BG, \mu_{l^4})$), the $\left(\sigma_{B\,\mathrm{O}(A,q),c(3\mathrm{Pin})}, \rho\right)$-module structure on F can be reduced to a (σ_{BG}, ρ_{BG})-module structure as in Proposition B.23. This in particular yields a theory on $\mathbf{Bord}_3^{BG}$ by Proposition B.14 which agrees with the theory obtained by trivializing the anomaly by Theorem 3.13(iii). The same is true when $3\mathrm{Pin}(A,q)$ is replaced by $3\mathrm{Spin}(A,q)$ and $f\colon G \to \mathrm{O}(A,q)$ is replaced by $f\colon G \to \mathrm{SO}(A,q)$.

Corollary 5.34. *Assuming Hypothesis Q for a* **BrFus***-compatible target $\mathcal{T}$ (Definition 5.30), there is an $H^3(B\,\mathrm{O}(n,n;\mathbb{F}_p), \mathbf{k}^\times)$-torsor of $\left(\sigma_{B\,\mathrm{O}(n,n;\mathbb{F}_p)}, \rho_{B\,\mathrm{O}(n,n;\mathbb{F}_p)}\right)$-module structures on $F = \sigma^3_{B\mathbb{F}_p^n}$. (Here $F = \sigma^3_{B\mathbb{F}_p^n}$ since both send the point to* $\mathbf{Vect}\left[\mathbb{F}_p^n\right]$*, by Example B.9(iii).)*

Proof. By Corollary 3.17(d), the trivialization defining the $\mathrm{O}(n,n;\mathbb{F}_p)$-theory in Corollary 5.29 (with underlying framed theory F) defines a $\left(\sigma_{B\,\mathrm{O}(n,n;\mathbb{F}_p)}, \rho_{B\,\mathrm{O}(n,n;\mathbb{F}_p)}\right)$-module structure on $F = \sigma^3_{B\mathbb{F}_p^n}$. □

Appendix A. TQFT and category theory

The axioms for functorial quantum field theory originated in [Ati88, Seg04]. In this work, we will use TQFT to refer to *fully-extended functorial topological quantum field theories*, which first appeared in [BD95]. These were further studied in [Lur09]. Other references include [Fre93, Wal06, Tel16, SP14, Kap10].

A.1. **The Cobordism Hypothesis.** The Cobordism Hypothesis was formulated by Baez-Dolan in [BD95]. A detailed sketch of the proof was given in [Lur09], and an approach for a proof using factorization homology is in [AF17]. A proof in two dimensions is in [SP09]. See [Fre13, SP14] for other references.

Let $d \in \mathbb{Z}^{\geq 0}$. Given a symmetric monoidal $(\infty, d+1)$-category $\mathcal{T}$, which will be the target for our theories, write $\mathcal{T}^\sim$ for the groupoid obtained by discarding all non-invertible morphisms at all levels. We will assume $\mathcal{T}$ has duals.[16] We will also assume that $\Omega^{d+1}\mathcal{T} \simeq \mathbf{k}$ for $\mathbf{k}$ an algebraically closed field of characteristic zero. Throughout the paper, we will write **Vect** for the $\mathbf{k}$-linear category of finite-dimensional vector spaces over $\mathbf{k}$.

Remark A.1. Besides having duals and satisfying $\Omega^{d+1}\mathcal{T} \simeq \mathbf{k}$, we will occasionally assume that $\mathcal{T}$ satisfies an additional hypothesis: When we discuss twisted module structures, in the sense of [FMT22] (as yet another avatar of anomalous theories), we will assume that the target $\mathcal{T}$ is sufficiently additive to support the construction of π-finite spaces in the sense of [FHLT10, §8] and [FMT22, §A] (this is Hypothesis Q).

The (framed) Cobordism Hypothesis, [Lur09, Theorem 2.4.6], asserts that, since $\mathcal{T}$ has duals, the functor given by evaluation on the point is an equivalence:

$$\mathrm{Fun}^{\otimes}\left(\mathbf{Bord}_{d+1}^{\mathrm{fr}}, \mathcal{T}\right) \xrightarrow{\sim} \mathcal{T}^\sim .$$

Remark A.2. The category $\mathrm{Fun}^{\otimes}(\mathbf{Bord}, \mathcal{T})$ consists of symmetric monoidal functors with morphisms given by strong natural transformations (in the sense of [JFS17]). A priori this is an $(\infty, d+1)$-category, but the statement of the Cobordism Hypothesis is that it is actually an $(\infty, 0)$-category, and in particular it is equivalent to the $(\infty, 0)$-category $\mathcal{T}^\sim$.

We will occasionally use the variant of $\mathrm{Fun}^{\otimes}(\mathbf{Bord}, \mathcal{T})$ which has the same objects, however the 1-morphisms are lax natural transformations. We will write this as $\mathrm{Fun}^{\mathrm{lax}}(\mathbf{Bord}, \mathcal{T})$. See Remark A.11.

[16] Otherwise replace $\mathcal{T}$ with the subcategory consisting of the fully-dualizable objects of $\mathcal{T}$.

A.1.1. *The Cobordism Hypothesis for X-theories.* There is a refined version of the Cobordism Hypothesis obtained as follows. We follow [Lur09]. Let X be a topological space with real rank $d+1$ vector bundle ζ, and let M be a manifold of dimension $m \leq d+1$. An (X,ζ)-structure on M consists of

(i) a continuous map $f\colon M \to X$, and
(ii) an isomorphism of bundles $TM \oplus \mathbb{R}^{d+1-m} \simeq f^*\zeta$.

Write $\mathbf{Bord}_{d+1}^{(X,\zeta)}$ for the category of bordisms equipped with an (X,ζ)-structure. This is [Lur09, Definition 2.4.17]. We will refer to symmetric monoidal functors out of $\mathbf{Bord}_{d+1}^{(X,\zeta)}$ as $(d+1)$-dimensional (X,ζ)-TQFTs, or just (X,ζ)-theories. The Cobordism Hypothesis for (X,ζ)-theories is [Lur09, Theorem 2.4.18]. It is the equivalence

$$\mathrm{Fun}^{\otimes}\left(\mathbf{Bord}_{d+1}^{(X,\zeta)}, \mathcal{T}\right) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{O}(d+1)}\left(\widetilde{X}, \mathcal{T}^{\sim}\right) \ , \tag{A.3}$$

where $\widetilde{X}$ is the associated principal $\mathrm{O}(d+1)$-bundle of orthonormal frames in ζ, and the right-hand side denotes $\mathrm{O}(d+1)$-equivariant maps, for the natural action of $\mathrm{O}(d+1)$ on $\mathcal{T}^{\sim}$ [Lur09, Theorem 2.4.18].

Let G be a topological group with a continuous group homomorphism to the orthogonal group $\chi\colon G \to \mathrm{O}(d+1)$. Particularly important cases of (X,ζ)-structures are given by $X = BG$ and $\zeta = \left(\mathbb{R}^{d+1} \times EG\right)/G$ defined by χ. In this case, we sometimes call theories defined on $\mathbf{Bord}_{d+1}^{BG}$ *G-theories*.

Example A.4. Let G be trivial and $\chi = \iota$ be the inclusion. A (BG,ζ_ι)-structure is a framing.

Example A.5. Let $G = \mathrm{SO}(d+1)$ and $\chi = \iota$ be the inclusion. A $(B\,\mathrm{SO}(d+1), \zeta_\iota)$-structure is an orientation.

We will write $\mathbf{Bord}_{d+1}^{X}$ for the category of bordisms with (X,ζ)-structure with ζ trivial. E.g. $X = BG$ and $\chi\colon G \to \mathrm{O}(d+1)$ trivial. In spite of fr being removed from the notation, one should think of $\mathbf{Bord}_{d+1}^{X}$ as consisting of framed bordisms with maps to X. We will refer to symmetric monoidal functors $\mathbf{Bord}_{d+1}^{X} \to \mathcal{T}$ as *X-TQFTs* (or just X-theories). These should be thought of as families of framed TQFTs over X. In this case, the Cobordism Hypothesis (A.3) reduces to:

$$\mathrm{Fun}^{\otimes}\left(\mathbf{Bord}_{d+1}^{X}, \mathcal{T}\right) \xrightarrow{\sim} \mathrm{Hom}\left(X, \mathcal{T}^{\sim}\right) \ . \tag{A.6}$$

We will also use (A.6) for once-categorified d-dimensional theories $\mathbf{Bord}_{d}^{X} \to \mathcal{T}$. Since $\mathbf{Bord}_{d}^{X}$ is an (∞,d)-category, such a functor factors through the underlying (∞,d)-category of $\mathcal{T}$, obtained by discarding the non-invertible $(d+1)$-morphisms. This has the same maximal ∞-groupoid $\mathcal{T}^{\sim}$, so (A.6) with $d+1$ replaced by d shows that these theories are again classified by functors $X \to \mathcal{T}^{\sim}$.

Example A.7. When $X = BG$ with $\chi = \mathrm{triv}\colon G \to \mathrm{O}(d+1)$ trivial, then the Cobordism Hypothesis reduces to [Lur09, Theorem 2.4.26]:

$$\mathrm{Fun}^{\otimes}\left(\mathbf{Bord}_{d+1}^{BG}, \mathcal{T}\right) \xrightarrow{\sim} \left(\mathcal{T}^{\sim}\right)^{hG}$$

where $(\mathcal{T})^{hG}$ denotes the homotopy fixed points of $\mathcal{T}$.

We will sometimes call symmetric monoidal functors $\mathbf{Bord}_{d+1}^{BG} \to \mathcal{T}$ *BG-theories*, or *G-theories*.

Endomorphisms of the trivial TQFT are equivalent to theories of one lower dimension. By the Cobordism Hypothesis this corresponds to the fact that the endomorphisms of the unit of the target form its looping, a category of one lower dimension. Proposition A.9 is this result for X-theories.

Lemma A.8. *If $\mathcal{A}$ and $\mathcal{B}$ are symmetric monoidal $(\infty, d+1)$-categories, then*

$$\Omega\,\mathrm{Fun}^{\mathrm{lax}}(\mathcal{A},\mathcal{B}) \simeq \mathrm{Fun}^{\otimes}(\mathcal{A}, \Omega\mathcal{B}) \ ,$$

where Ω is taken at the trivial functor (cf. [JFS17, Theorem 7.4]*).*

Proof. By definition [JFS17, Definition 1.3], a lax natural transformation from the trivial functor to itself is a functor $\mathcal{A} \to \mathcal{B}^{\downarrow}$ whose composites with the source and target functors $s, t \colon \mathcal{B}^{\downarrow} \to \mathcal{B}$ are trivial, i.e. a functor to

$$\mathcal{B}[0] \overset{h}{\underset{\mathcal{B}}{\times}} \mathcal{B}^{\downarrow} \overset{h}{\underset{\mathcal{B}}{\times}} \mathcal{B}[0] \simeq \Omega\mathcal{B}$$

(see [JFS17, Proposition 6.12], as in the proof of Theorem 2.4). □

Proposition A.9. *If 1_X is the trivial once-categorified theory $\mathbf{Bord}_d^X \to \mathcal{T}$, then*

$$\mathrm{Fun}^{\otimes}\left(\mathbf{Bord}_d^X, \Omega\mathcal{T}\right) \simeq \mathrm{End}_{\mathrm{Fun}^{\mathrm{lax}}}(1_X) \ ,$$

where $\Omega\mathcal{T} \coloneqq \mathrm{End}_{\mathcal{T}}(1)$ denotes the looping of $\mathcal{T}$.

Proof. This follows from Lemma A.8 for $\mathcal{A} = \mathbf{Bord}_d^X$ and $\mathcal{B} = \mathcal{T}$. □

A.2. **Relative theories.** A symmetric monoidal functor

$$\alpha \colon \mathbf{Bord}_d^X \to \mathcal{T}$$

is a *once-categorified d-dimensional X-TQFT*, where $\mathcal{T}$ is the fixed target from the beginning of Section A.1. Let

$$1_X \colon \mathbf{Bord}_d^X \to \mathcal{T}$$

denote the trivial once-categorified d-dimensional X-TQFT. Recall the notion of a relative theory [FT14]. These are also called *twisted theories* [ST11, JFS17].

Definition A.10. A theory defined right-relative to α is a lax natural transformation (in the sense of [JFS17])

$$\alpha \to 1 \ .$$

A theory defined left-relative to α is a lax natural transformation

$$1 \to \alpha \ .$$

Remark A.11. Recall the definition of lax (resp. oplax) natural transformations from [JFS17]. Consider the arrow categories $\mathcal{T}^{\downarrow}$ and $\mathcal{T}^{\rightarrow}$, and the source and target functors $s, t \colon \mathcal{T}^{*} \to \mathcal{T}$ for $* = \downarrow, \rightarrow$. Following [JFS17], a lax (resp. oplax) natural transformation $\alpha \to 1$ is a functor

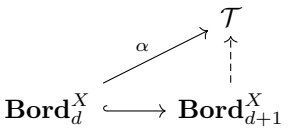
$$F_\alpha \colon \mathbf{Bord}_d^X \to \mathcal{T}^{\downarrow} \qquad \left(\text{resp. } F_\alpha \colon \mathbf{Bord}_d^X \to \mathcal{T}^{\rightarrow}\right) \tag{A.12}$$

such that $s \circ F_\alpha = \alpha$, and $t \circ F_\alpha = 1$.

Throughout, we will use the lax version, as written in the above definition of relative theories. The reason we use the lax version, as noted in [JFS17, Example 7.3], is that the *lax* natural transformations from the trivial theory to itself consist of theories of dimension lower ([JFS17, Theorem 7.4]) whereas the same is not true when lax is replaced with oplax. We need the analogous result for X-theories (Proposition A.9), in particular to establish a trivialized anomalous theory $1 \xrightarrow{\sim} \alpha \to 1$ as an honest theory of one dimension lower.

Also noted in [JFS17, Example 7.3], is the fact that oplax natural transformations are "elements" $F_\sigma(M) : 1 \to \alpha(M)$ for M a *closed* bordism of any codimension, which is for example the point of view taken in [FT14]. Oplax natural transformations are also used in [FT21]. Besides Proposition A.9, and the results depending on it, the remaining constructions and facts in this paper hold for the oplax version, obtained by replacing all lax natural transformations with oplax ones, and replacing $\mathcal{T}^{\downarrow}$ with $\mathcal{T}^{\rightarrow}$.

Remark A.13. Often the once-categorified d-dimensional theory α extends to a $(d+1)$-dimensional theory:



In this case, the relative theory $\alpha \to 1$ is upgraded to what is called a *boundary theory*. Boundary theories are defined as functors out of the extended bordism category $\mathbf{Bord}_{d+1}^{X,\partial}$, described in [Lur09, §4.3]. See also [Ste24]. The connection with the notion defined here is made in [JFS17, Theorem 7.15].

A.3. **Invertible theories.** Let $\left|\mathbf{Bord}_{d+1}^X\right|$ denote the completion, i.e. the $(\infty,0)$-category obtained by manually inverting all the arrows, and let $\mathcal{T}^\times$ denote the underlying sub-groupoid, i.e. the $(\infty,0)$-category obtained by discarding all non-invertible objects and non-invertible morphisms at all levels.

A TQFT $\alpha\colon \mathbf{Bord}_{d+1}^X \to \mathcal{T}$ is *invertible* if it factors as:

(A.14)
$$\begin{array}{ccc} \mathbf{Bord}_{d+1}^X & \xrightarrow{\alpha} & \mathcal{T} \\ \downarrow & & \uparrow \\ \left|\mathbf{Bord}_{d+1}^X\right| & \xrightarrow{\widetilde{\alpha}} & \mathcal{T}^\times \end{array}$$

I.e. it assigns all objects and morphisms in the source to invertible objects and morphisms in the target.

Remark A.15. Note that (A.14) reduces the functor α to a map of spectra $\widetilde{\alpha}$, since both $\left|\mathbf{Bord}_{d+1}^X\right|$ and $\mathcal{T}^\times$ are infinite loop spaces on account of their symmetric monoidal structures [FH21]. This allows us to study it as a cohomology class (in the theory determined, via Brown representability, by the target) on the source.

Appendix B. Topological symmetry

B.1. **TQFTs associated to π-finite spaces.** Let X be a space (i.e. higher groupoid) which is (connected, pointed, and) π-finite[17]. There is a recipe for constructing a TQFT using X, which was introduced in [Kon88] and studied further in [Qui95, Tur10]. We will consider the fully local case introduced in [Fre94] and studied in [FHLT10, §3,8] and [FMT22, §A].

In the 1-dimensional case, the finite path-integral has mathematical rigorous foundations in the ∞-categorical setting [Har20]. Related work in a higher setting is the subject of an upcoming work of Claudia Scheimbauer and Tashi Walde.

B.1.1. *The summation map.* We will proceed heuristically, following [FHLT10, §3] and [FMT22, §A.2], to fix notation and describe expectations which will eventually be stated and assumed in Hypothesis Q.

Let $\mathbf{Fam}_{d+1}$ denote the category with objects finite $(d+1)$-groupoids, 1-morphisms given by correspondences of π-finite spaces, 2-morphisms given by correspondences of correspondences, and so on until level $(d+1)$. (Two $(d+1)$-morphisms are regarded as identical if they are equivalent.)

Let $\mathcal{T}$ be the symmetric monoidal target with duals, fixed in Section A.1. Let Y be an object of $\mathbf{Fam}_{d+1}$. A local system on Y valued in $\mathcal{T}$ is a functor $Y \to \mathcal{T}$. Write $\mathbf{Fam}_{d+1}(\mathcal{T})$ for the category of π-finite spaces equipped with a local system valued in $\mathcal{T}$, with morphisms given by correspondences equipped with morphisms between the pulled-back local systems.

[17] This means X has finitely many nonzero homotopy groups, each of which is finite.

For example, $\mathbf{Fam}_{d+1}\left(B^{d+1}\mathbf{k}^\times\right)$ has objects given by pairs (Y,c), where c is a cocycle

$$c\colon Y \to B^{d+1}\mathbf{k}^\times$$

representing a class in $H^{d+1}\left(Y,\mathbf{k}^\times\right)$. A morphism is a correspondence:

$$\begin{array}{ccc} & (E,\mu) & \\ {\scriptstyle p_1}\swarrow & & \searrow{\scriptstyle p_2} \\ (Y_1,c_1) & & (Y_2,c_2) \end{array} \tag{B.1}$$

where $\mu\colon E \to B^d\mathbf{k}^\times$ satisfies

$$d\mu = \left(p_1^* c_1\right)^{-1} \cdot \left(p_2^* c_2\right) \ . \tag{B.2}$$

2-morphisms are correspondences between correspondences with a similar condition on the cocycles, and so on to define morphisms up to level $(d+1)$.

Recall we assumed $\Omega^{d+1}\mathcal{T} \simeq \mathbf{k}$. Therefore there is a natural functor: $B^{d+1}\mathbf{k}^\times \to \mathcal{T}$ inducing a functor:

$$\mathbf{Fam}_{d+1}\left(B^{d+1}\mathbf{k}^\times\right) \to \mathbf{Fam}_{d+1}\left(\mathcal{T}\right) \ . \tag{B.3}$$

We assume that the following holds.

Hypothesis Q. *For $\mathcal{T}$ appropriately additive*[18]*, there is a "quantization"*[19] *functor*

$$\mathrm{Sum}_{d+1}\colon \mathbf{Fam}_{d+1}\left(B^{d+1}\mathbf{k}^\times\right) \to \mathcal{T} \tag{B.4}$$

such that there is an invertible natural transformation between

$$(X,c) \mapsto \mathrm{Hom}_{\mathcal{T}}\left(1, \mathrm{Sum}_{d+1}\left(X,c\right)\right)$$

and

$$(X,c) \mapsto \mathrm{Hom}_{c}\left(\mathrm{hofib}\left(c\right), \Omega\mathcal{T}\right)$$

viewed as functors out of $\mathbf{Fam}_{d+1}\left(B^{d+1}\mathbf{k}^\times\right)$.

Here $\mathrm{hofib}\left(c\right) \to X$ is a principal $B^d\mathbf{k}^\times$-bundle, and $B^d\mathbf{k}^\times$ acts on $\Omega\mathcal{T}^\sim$ by tensoring with scalars (via the inclusion of scalars $B^d\mathbf{k}^\times \to \left(\Omega\mathcal{T}\right)^\times$, using $\Omega^{d+1}\mathcal{T} \simeq \mathbf{k}$). We write $\mathrm{Hom}_c\left(\mathrm{hofib}\left(c\right), \Omega\mathcal{T}\right)$ for the space of $B^d\mathbf{k}^\times$-equivariant maps $\mathrm{hofib}\left(c\right) \to \Omega\mathcal{T}^\sim$, i.e. maps whose restriction to each fiber $B^d\mathbf{k}^\times$ is the canonical inclusion of scalars, together with the data exhibiting them as equivariant. This is in general smaller than the space $\mathrm{Hom}\left(\mathrm{hofib}\left(c\right), \Omega\mathcal{T}\right)$ of all functors.

Remark B.5. Unpacking the existence of the natural transformation in Hypothesis Q, we see that it ensures that we have equivalences

$$\mathrm{Hom}_{\mathcal{T}}\left(1, \mathrm{Sum}_{d+1}\left(X,c\right)\right) \simeq \mathrm{Hom}_{c}\left(\mathrm{hofib}\left(c\right), \Omega\mathcal{T}\right) \tag{B.6}$$

for all objects (X,c) of $\mathbf{Fam}_{d+1}\left(B^{d+1}\mathbf{k}^\times\right)$, and given a morphism from (X_1,c_1) to (X_2,c_2) in $\mathbf{Fam}_{d+1}\left(B^{d+1}\mathbf{k}^\times\right)$, we have a commuting diagram:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{T}}\left(1, \mathrm{Sum}_{d+1}\left(X_1,c_1\right)\right) & \simeq & \mathrm{Hom}_{c_1}\left(\mathrm{hofib}\left(c_1\right), \Omega\mathcal{T}\right) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathcal{T}}\left(1, \mathrm{Sum}_{d+1}\left(X_2,c_2\right)\right) & \simeq & \mathrm{Hom}_{c_2}\left(\mathrm{hofib}\left(c_2\right), \Omega\mathcal{T}\right) \end{array} \tag{B.7}$$

This generalizes the classical fact about modules over the group algebra and G-representations, as in Example B.9 (ii).

[18] As is explained in [FHLT10, §3], for X an ordinary groupoid, then $\mathcal{T}$ must be additive in the sense that the colimit $\varinjlim_{x\in X} c\left(x\right)$ in $\mathcal{T}$ exists, and agrees with the limit $\varprojlim_{x\in X} c\left(x\right)$, where we are regarding c as defining a $\mathcal{T}$-valued local system on X by (B.3).

[19] See [FMT22, Remark A.7.1] where it is explained how $\mathrm{Sum}_{d+1}\left(X\right)$ (and therefore σ_X^{d+1}) can be obtained by integrating over fluctuating fields.

Remark B.8. In [FHLT10, §8.2] the map Sum_n is constructed at the level of objects and shown to be a functor up to 2-morphisms. In particular, the cases in Example B.9 are worked out in [FHLT10, §8.1,8.3].

Example B.9. (i) Let $\mathcal{T} = \mathbf{Vect}$. Then $\mathrm{Sum}_1(X) = \mathbf{k}(\pi_0 X)$ is the vector space of $\mathbf{k}$-valued functions on $\pi_0 X$. Hypothesis Q is satisfied: the natural transformation (B.6) is given by the following equivalences, whose naturality is straightforward to check:

$$\mathrm{Hom}_{\mathbf{Vect}}(1, \mathrm{Sum}_1(X)) \simeq \mathrm{Hom}(X, \mathbf{k}) \simeq \mathrm{Hom}(\pi_0 X, \mathbf{k}) \ .$$

(ii) Set $X = BG$ for a finite group G, and let $\mathcal{T} = \mathbf{Alg}$ be the Morita 2-category of algebras. Let $\mathrm{Sum}_2(BG) = \mathbf{k}[G]$ be the group algebra. Hypothesis Q is satisfied: the natural transformation (B.6) is given by the following equivalences, whose naturality is straightforward to check:

$$\mathrm{Hom}_{\mathbf{Alg}}(1, \mathrm{Sum}_2(BG)) = \mathbf{k}[G]\text{-}\mathbf{mod} \simeq \mathbf{Rep}(G) \simeq \mathrm{Hom}(BG, \mathbf{Vect}) \ .$$

We can equip BG with a 2-cocycle c (i.e. a group 2-cocycle) which twists the convolution structure on the group algebra, resulting in $\mathrm{Sum}_2(BG, c)$.

(iii) Set $X = BG$. Let $d = 2$, and take $\mathcal{T}$ to be the Morita 3-category of fusion categories. Then $\mathrm{Sum}_3(BG)$ is $\mathbf{Vect}[G]$, the fusion category of vector bundles on G with convolution. We can equip BG with a 3-cocycle ω which twists the fusion structure on $\mathbf{Vect}[G]$, yielding $\sigma^3_{BG,\omega}(*)$. This is [FMT22, Example A.65].

(iv) Set $X = B^2G$ for a finite abelian group G. Let $d = 3$ and take $\mathcal{T}$ to be the Morita 4-category of braided monoidal categories, **BrFus** (see Section 4.1). Consider a cocycle $c\colon B^2G \to B^4\mathbf{k}^\times$. It is a theorem of Eilenberg-Mac Lane [EML54, Theorem 26.1] that cohomology classes in $H^4(B^2G, \mathbf{k}^\times)$ correspond to quadratic forms $G \to \mathbf{k}^\times$. Then $\mathrm{Sum}_4(B^2G)$ is $\mathbf{Vect}[G]$ with convolution, and with associator and braiding (4.19) given by an abelian 3-cocycle (τ, b) representing $[c]$ (see Section 4.3.2).

(v) Let $d = 3$ and $\mathcal{T}$ be a 4-category of monoidal 2-categories (e.g. $2\mathbf{Fus}$ [DR18]). For any finite group G, the fusion 2-category $\mathrm{Sum}_4(BG)$ is the collection of G-graded 2-vector spaces [DR18, Construction 2.1.13].

As far as the author is aware, dualizability in the Morita 4-category of monoidal 2-categories has not been extensively studied. However this fusion 2-category is expected to define a fully extended 4-dimensional TQFT [DR18]. See Section 5.8 where we discuss this example in more detail.

B.1.2. *TQFTs from groupoids.* The upshot of assuming Hypothesis Q is that, for a fixed object (X, c) of $\mathbf{Fam}_{d+1}(B^{d+1}\mathbf{k}^\times)$, (B.4) can be composed with the mapping space functor[20] to obtain the theory

$$\sigma^{d+1}_{X,c}\colon \mathbf{Bord}^{\mathrm{fr}}_{d+1} \xrightarrow{\pi_{\leq d+1}\mathrm{Map}(-,X)} \mathbf{Fam}_{d+1}(B^{d+1}\mathbf{k}^\times) \xrightarrow{\mathrm{Sum}_{d+1}} \mathcal{T} \ .$$

Remark B.10. In [FHLT10, §3,8], the theories σ^{d+1}_X are studied for $\mathcal{T} = \mathbf{Alg}[d]$ the Morita $(d+1)$-category of "d-algebras", discussed in [FHLT10, §7].

Remark B.11. As is remarked in [FMT22, §A] and [FHLT10, §3,8], $\sigma^{d+1}_{X,c}$ can be upgraded to an oriented theory, and if c is trivial then it can even be upgraded to an unoriented theory. We will work with the underlying framed theories in this paper.

Proposition B.12 ([FMT22])**.** *Assuming Hypothesis Q, a morphism from (X_1, c_1) to (X_2, c_2) in $\mathbf{Fam}_{d+1}(B^{d+1}\mathbf{k}^\times)$ (i.e. correspondence as in (B.1)) induces a bimodule (i.e. domain wall)*

$$\sigma^{d+1}_{X_1,c_1} \to \sigma^{d+1}_{X_2,c_2} \ .$$

[20] Note that (B.3) allows us to construct a $\mathcal{T}$-valued local system from the cocycle c.

Proof. By the Cobordism Hypothesis, the functor Sum_{d+1} in Hypothesis Q provides a functor:

$$\begin{aligned} \mathbf{Fam}_{d+1}\left(B^{d+1}\mathbf{k}^{\times}\right) &\longrightarrow \mathrm{Fun}^{\otimes}\left(\mathbf{Bord}_{d+1}^{\mathrm{fr}}, \mathcal{T}\right) \\ (X,c) &\longmapsto \sigma_{X,c}^{d+1} \end{aligned}$$

Proposition B.12 follows from the fact that the correspondence is a morphism in the category $\mathbf{Fam}_{d+1}\left(B^{d+1}\mathbf{k}^{\times}\right)$. □

Remark B.13. Proposition B.12 can be shown directly (i.e. without the Cobordism Hypothesis) since a correspondence of spaces $X \leftarrow C \rightarrow Y$ defines a correspondences of mapping spaces $\mathrm{Map}(M,X) \leftarrow \mathrm{Map}(M,C) \rightarrow \mathrm{Map}(M,Y)$, for any bordism M. This is the perspective taken in [FMT22].

Recall the motivation in Remark B.5 for the existence of the natural transformation in Hypothesis Q. Proposition B.14 writes this in terms of the theories σ_X^{d+1}.

Proposition B.14. *Assuming Hypothesis Q, every boundary theory for $\sigma_{X,c}^{d+1}$ is classified by a symmetric monoidal functor:*

$$\mathbf{Bord}_d^{\mathrm{hofib}(c)} \rightarrow \Omega\mathcal{T}$$

whose classifying map is equipped with the structure of a $B^d\mathbf{k}^{\times}$-equivariant map, i.e. lifts to $\mathrm{Hom}_c\left(\mathrm{hofib}\left(c\right), \Omega\mathcal{T}\right)$. If c is trivial, then the boundary theories are classified by symmetric monoidal functors:

$$\mathbf{Bord}_d^X \rightarrow \Omega\mathcal{T}\ .$$

Proof. By [JFS17, Theorem 7.15], the boundary theories for $\sigma_{X,c}^{d+1}$ are classified by

$$\mathrm{Hom}_{\mathcal{T}}\left(1, \sigma_{X,c}^{d+1}\left(*\right)\right)$$

which is the LHS of (B.6). Similarly, by the Cobordism Hypothesis (A.6), the symmetric monoidal functors from $\mathbf{Bord}_d^{\mathrm{hofib}(c)}$ to $\Omega\mathcal{T}$ are classified by $\mathrm{Hom}\left(\mathrm{hofib}\left(c\right), \Omega\mathcal{T}\right)$, and the RHS of (B.6) is Hom_c, which maps to it by forgetting the equivariance. If c is trivial, then a trivialization identifies $\mathrm{hofib}\left(c\right) \simeq X \times B^d\mathbf{k}^{\times}$ as principal $B^d\mathbf{k}^{\times}$-bundles. Equivariant maps out of the trivial bundle $X \times B^d\mathbf{k}^{\times}$ are the same as maps out of X, so the RHS of (B.6) is equivalent to $\mathrm{Hom}\left(X, \Omega\mathcal{T}^{\sim}\right)$, i.e. by (A.6) to functors $\mathbf{Bord}_d^X \rightarrow \Omega\mathcal{T}$. □

B.2. **Module structures.** We summarize the material used in our construction from [FMT22]. See [FMT22, §3] for a more detailed discussion of these definitions.

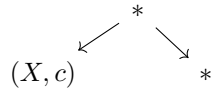
Let

$$\sigma\colon \mathbf{Bord}_{d+1}^{\mathrm{fr}} \rightarrow \mathcal{T}$$

be a $(d+1)$-dimensional TQFT valued in the fixed target $\mathcal{T}$ from the beginning of Section A.1. Recall the notion of a boundary theory from Remark A.13. Recall the following definition from [FMT22].

Definition B.15. A *d-dimensional quiche* is a pair (σ, ρ) in which ρ is a right topological boundary theory (or *right σ-module*), which we will write as $\rho\colon \sigma \rightarrow 1$.

Remark B.16. All of the quiches considered in this paper will be of the form $(\sigma_{X,c}, \rho_{X,c})$, where $\sigma_{X,c}$ is the theory associated to (X,c) as in Section B.1, and the (right) boundary theory $\rho_{X,c}$ is the natural transformation induced by the correspondence diagram:



as in Proposition B.12. Given a pointed space X, we will always write $\rho_{X,c}$ for this boundary theory.

Recall, from [JFS17, Corollary 7.7], there is an equivalence of $(\infty, d+1)$-categories between the following.

(1) The $(\infty, d+1)$-category $\mathrm{Fun}^{\mathrm{lax}}\left(\mathbf{Bord}_{d+1}^{\mathrm{fr}}, \mathcal{T}\right)$ of framed TQFTs with 1-morphisms given by relative[21] field theories between them, and j-morphisms given by what Johnson-Freyd-Scheimbauer call j-times-twisted field theories.
(2) The $(\infty, d+1)$-subcategory of $\mathcal{T}$ consisting of fully dualizable objects, and with morphisms which are $(d+1)$-times left-adjunctible.

Given any quiche (σ, ρ), the boundary theory ρ is equivalent to a relative theory $\tau_{\leq d}\sigma \to 1$ by [JFS17, Theorem 7.15], and is classified by a 1-morphism

$$|\rho| : \sigma(*) \to 1$$

in $\mathcal{T}$.

Definition B.17. A morphism between quiches $(\sigma, \rho) \to (\sigma', \rho')$ is a 1-morphism $\delta\colon \sigma(*) \to \sigma'(*)$ in $\mathcal{T}$ along with an invertible 2-morphism $s\colon |\rho| \Rightarrow |\rho'| \circ \delta$.

A quiche is an abstract symmetry datum, in the same sense as an algebra. The following definition, from [FMT22], is the analogue of a module, i.e. a realization of the quiche as symmetries of a given theory.

Definition B.18. Let (σ, ρ) be a d-dimensional quiche, and let F be a d-dimensional TQFT. A (σ, ρ)-module structure on F is the pair (F_σ, θ) where F_σ is a (left) boundary theory $F_\sigma\colon 1 \to \sigma$ which is equipped with an isomorphism of d-dimensional theories:

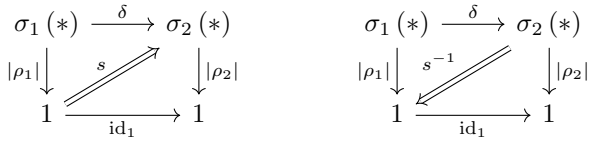
$$\theta\colon \rho \otimes_\sigma F_\sigma \xrightarrow{\sim} F\ .$$

B.3. **Reduction of topological symmetry.** Let X be a (pointed, connected) π-finite space, and consider a cocycle $c\colon X \to B^{d+1}\mathbf{k}^\times$. Assuming Hypothesis Q, and given a $\left(\sigma_{X,c}^{d+1}, \rho_{X,c}\right)$-module structure on a theory F, we might wonder what extra data is needed to "reduce" this to a $\left(\sigma_X^{d+1}, \rho_X\right)$-module structure on F. Theorem B.19, which holds more generally, will tell us that a trivialization of c is sufficient to perform such an operation, which we will define to be a *reduction* of topological symmetry in Definition B.24.

Recall the analogy in [FMT22] between modules over an algebra (or linear representations of a Lie group) and field theories. Under this analogy, Theorem B.19 is the analogue of the fact that a bimodule induces a functor between the categories of modules.

Theorem B.19. *Assume Hypothesis Q. Given a (σ_1, ρ_1)-module structure on a d-dimensional theory F, then a morphism of quiches (Definition B.17) from (σ_1, ρ_1) to (σ_2, ρ_2) canonically defines a (σ_2, ρ_2)-module structure on F.*

Proof. Write the given (σ_1, ρ_1)-module structure on F as (F_{σ_1}, θ_1) (as in Section B.2). A morphism of quiches (Definition B.17) consists of an invertible 2-morphism:



[21] Recall from Remark A.11 we are using lax natural transformations/relative theories in this paper.

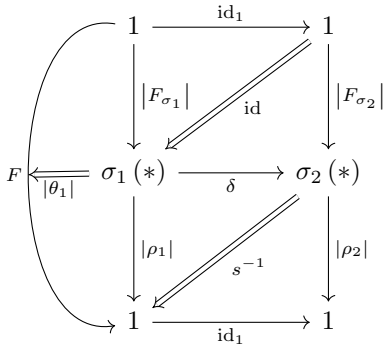
The morphism δ induces a (σ_2, σ_1)-bimodule, i.e. a domain wall $D\colon \sigma_1 \to \sigma_2$. Define the (σ_2, ρ_2)-module structure (F_{σ_2}, θ_2) as follows. Define the boundary theory F_{σ_2} to be

$$F_{\sigma_2} = D \otimes_{\sigma_1} F_{\sigma_1} \ .$$

Define the equivalence

$$\theta_2\colon \rho_2 \otimes_{\sigma_2} F_{\sigma_2} = \rho_2 \otimes_{\sigma_2} D \otimes_{\sigma_1} F_{\sigma_1} \xrightarrow{\sim} F$$

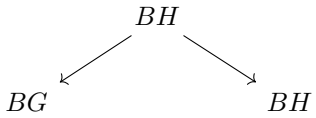
to be induced by the following composition:



where $|\theta_1|$ is the 2-morphism in $\mathcal{T}$ classifying the equivalence:

$$\theta_1\colon \rho_1 \otimes_{\sigma_1} F_{\sigma_1} \xrightarrow{\sim} F \ . \qquad \square$$

Example B.20. The inclusion of a subgroup $H \subset G$ defines a correspondence



which induces a morphism of quiches

$$\left(\sigma_{BG}^{d+1}, \rho_{BG}\right) \to \left(\sigma_{BH}^{d+1}, \rho_{BH}\right) \ .$$

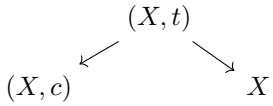
If a framed d-dimensional TQFT F has a $\left(\sigma_{BG}^{d+1}, \rho_{BG}\right)$-module structure, then the induced $\left(\sigma_{BH}^{d+1}, \rho_{BH}\right)$-module structure from Theorem B.19 is the restriction of the original module structure along the inclusion of H. In Example B.9 (ii) this is literally the restriction of a G-representation to an H-representation.

Now we return to X a π-finite space with $c\colon X \to B^{d+1}\mathbf{k}^\times$.

Proposition B.21. *A trivialization of the class c induces a $\left(\sigma_X^{d+1}, \rho_X\right)$-module structure on any theory with a $\left(\sigma_{X,c}^{d+1}, \rho_{X,c}\right)$-module structure.*

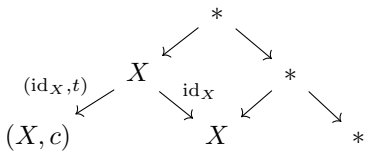
Proof. A trivialization t of c determines a correspondence as in (B.1):

(B.22)



By Proposition B.12, this induces a domain wall D from $\sigma_{X,c}^{d+1}$ to σ_X^{d+1}, i.e. a 1-morphism $\delta := |D| : \sigma_{X,c}^{d+1}(*) \to \sigma_X^{d+1}(*)$ in $\mathcal{T}$. The composite of (B.22) with the correspondence inducing ρ_X

(Remark B.16) is the following composition of correspondences:

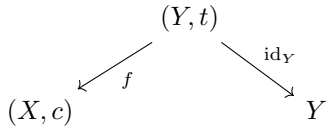


This is just the pointing of (X, c) (choosing t compatibly with the basepoint), i.e. it induces $\rho_{X,c}$. Since Sum_{d+1} is a functor (Hypothesis Q), this gives an invertible 2-morphism $s\colon |\rho_{X,c}| \Rightarrow |\rho_X| \circ \delta$. Hence (δ, s) is a morphism of quiches (Definition B.17) from $\left(\sigma^{d+1}_{X,c}, \rho_{X,c}\right)$ to $\left(\sigma^{d+1}_X, \rho_X\right)$, and Theorem B.19 implies the result. □

We can pull c back along a map $f\colon Y \to X$ so, even if it is not trivializable on X, it may be on Y.

Proposition B.23. *A trivialization of f^*c induces a $\left(\sigma^{d+1}_Y, \rho_Y\right)$-module structure on any theory with a $\left(\sigma^{d+1}_{X,c}, \rho_{X,c}\right)$-module structure.*

Proof. A trivialization t of f^*c determines a correspondence



and the rest of the proof is the same as that of Proposition B.21. □

Definition B.24. Let $f\colon Y \to X$ be a map between π-finite spaces. Given a trivialization of $f^*c\colon Y \to B^{d+1}\mathbf{k}^\times$, we will refer to the induced module structure from Proposition B.23 as a *reduction to* Y of the original module structure.

Example B.25. Let $\widetilde{X} = B\widetilde{G} \to BG = X$ for a group extension $\widetilde{G} \to G$. I.e. $d = 1$, and the class c classifies this central extension of G as usual. Theories with a $\left(\sigma_{B\widetilde{G}}, \rho_{B\widetilde{G}}\right)$-module structure are the same as representations of $\widetilde{G}$, i.e. projective representations of G with projectivity cocycle c. Proposition B.23 then says that splittings of $\widetilde{G}$ over $H \to G$ determine reductions of the linear representations of $\widetilde{G}$ to linear representation of H, as usual. See Example 3.9 and [Fre23, §1].

Statements and Declarations

Competing interests. The author has no relevant financial or non-financial interests to disclose.

Availability of data and materials. The author declares that the data supporting the findings of this study are available within the paper.

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