

INTEGRAL TRANSFORMS FOR FINITE GAUGE THEORIES

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ABSTRACT. Assuming the existence of a quantization functor for π -finite spaces out of a higher category of decorated spans, we study two kinds of equivariance of the resulting finite gauge theories. Symmetries of a given polarization/Lagrangian define semiclassical symmetries, which quantize to symmetries of the theory; the obstruction to making these linear is described by the Serre spectral sequence, and in dimension three, for untwisted theories, the resulting action is canonical and linear. On the other hand, symmetries of the entire phase space a priori only induce projective symmetries, with an invertible once-categorified theory, the anomaly theory, encoding the projectivity.

For three-dimensional finite abelian gauge theories we introduce a twice-categorified analogue of Blattner-Kostant-Sternberg kernels, and show that the associated Fourier-type integral transform realizes, up to isomorphism, the elements of $O(L \oplus L^\vee)$ exchanging L and $L^\vee$ in the canonical projective action of this orthogonal group. This establishes an analogy between certain instances of the π -finite quantization procedure considered herein and the geometric quantization of a symplectic vector space.

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1. INTRODUCTION

Geometric quantization relates (semi)classical states to quantum ones. Namely, the geometric quantization of a polarized symplectic vector space V consists of some sort of “linearization of half of the variables”. For instance, given a polarization $V \simeq T^*\ell$, one model for the Hilbert space of states is $L^2(\ell)$. This space of states has the curious property that, even though it is most naturally thought of as associated to the symplectic vector space V , it is only *projectively* equivariant for the natural symplectic group of symmetries.

The traditional way of identifying this projectivity is by constructing a natural integral transform T_g associated to each symplectic group element g , using what are called Blattner-Kostant-Sternberg (BKS) kernels [LV80, Pol03, BW97]. E.g. when g is the symplectic matrix sending the position coordinates in V to the momentum coordinates, the associated integral transform is the *Fourier transform*. The integral transforms don’t quite compose coherently:

$$T_g \circ T_h = c(g, h) T_{gh}$$

for some scalar $c(g, h) \in \mathrm{U}(1)$. This defines a 2-cocycle, and the extension classified by $[c] \in H^2(\mathrm{Sp}(T^*\ell), \mathrm{U}(1))$ (in measurable group cohomology) is essentially the metaplectic group.

In summary: Geometric quantization is only *projectively* a functor out of the appropriate category of phase spaces. On the other hand, it is manifestly (linearly) functorial out of *polarized* phase spaces.

We study an analogue of this projective (resp. linear) equivariance with respect to automorphisms of the full phase space (resp. of the Lagrangian) in the setting of quantization of π -finite groupoids in dimension three [Kon88, Qui95, Tur10, Fre94, FHLT10, FMT22]. Since a full treatment has not appeared in the literature, we assume a certain [Hypothesis Q](#) (and the cobordism hypothesis) which enables us to consider the *quantization* of a π -finite space, possibly decorated by a cocycle τ representing a class in $H^d(X)$, which is a d -dimensional TQFT $\sigma_{X, \tau}^d$. For example, when X is the classifying space of a finite group we obtain Dijkgraaf-Witten theory.

Under the analogy considered herein, the analogue of the symplectic vector space $V = T^*\ell$ is $A = L \oplus L^\vee$, with “Lagrangian” the finite subgroup L . The symplectic form is replaced by a quadratic¹ form on A given by evaluation. The analogue of the geometric quantization itself is the Dijkgraaf-Witten theory for L , written σ_{BL}^3 .

In any dimension, several different avatars of a “projective action” on a TQFT are shown to be equivalent in [VD23, §3.3]: Projective representations of $\mathbf{Bord}_d^{BG}$ are equivalent to modules over the quiche given by $(d+1)$ -dimensional pure topological gauge theory twisted by a representative for the projectivity class. This is the main upshot of [Hypothesis Q](#), as is spelled out here in [Proposition 2.2](#).

Restricting to dimension three, it is shown in [VD23, §5] that orthogonal automorphisms of an odd-order metric group A induce canonical projective automorphisms of the theory σ_{BL}^3 , which

¹There is a discrepancy between the two sides of the analogy: The symplectic structure is replaced by an orthogonal one. This is a version of the well-known analogy between the Weil representation and spinors [Del99]. Accordingly, the projective action of the orthogonal group considered here is in some respects closer to the spin representation than to the Weil representation; see [VD23, §4.7].

a priori need not linearize. (For L an elementary abelian p -group with p odd, the obstruction vanishes [EG18, Theorem 6.1]; see [Example 3](#).) Equivalently, certain 3-group extensions (defined in [VD23, §4.5]) of $O(A, \text{ev})$ act linearly on σ_{BL}^3 . In other words,² **twice-categorified geometric quantization of finite abelian odd-order metric groups is projectively equivariant with respect to the orthogonal group**. This is analogous to the linear action of the extension Spin of SO on spinors, as is explained in [VD23, §4.7].

This work spells out this canonical projective action in terms of “topological symmetries” [FMT22]. Using the classification of bimodule categories over pointed fusion categories [ENO10], we identify a semiclassical automorphism (i.e. a correspondence/span) realizing the action of each group element, up to isomorphism ([Section 5.4](#)). For the elements exchanging L and $L^\vee$, this correspondence is decorated by a canonical 2-cocycle on A , which plays an analogous role to that played by the BKS kernels ([Section 5.1](#)). The main theorems concerning this projective action ([Theorems 5.1](#) and [5.2](#)) can be summarized as follows ([Section 1.4](#) is dedicated to a more detailed overview):

Theorem. *The projective action of $O(A = L \oplus L^\vee)$ on σ_{BL}^3 can be described, up to isomorphism, via explicit bimodule categories. Explicitly, the order 2 element $T_\sigma \in O(A)$ determined by a symmetric isomorphism $\sigma: L \simeq L^\vee$, which exchanges L with $L^\vee$, acts on σ_{BL}^3 via the Fourier-type transform defined in [Definition 5.1](#).*

In contrast to the full orthogonal group, we show that if we only ask for automorphisms of any fixed Lagrangian L in A , then this induces a fully coherent linear action on the theory σ_{BL}^3 . In [Section 4](#) we also extend this result to the case of possibly nonabelian L , twisted by a 3-cocycle.

The quadratic form on A defines both this 2-cocycle on A (analogue of BKS kernel, as detailed in [Section 5.1](#)) and a class in $H^4(B^2A, \mathbf{k}^\times)$ [EM46]. Choosing a 4-cocycle representative of the latter, the quantization of A twisted by this 4-cocycle is the so-called Crane-Yetter theory sending the point to the category of vector spaces graded by A , with braiding determined by the quadratic form. This famously encodes the framing anomaly of Reshetikhin-Turaev theory: The regular bimodule over the braided category defines a relative/twisted/anomalous 3-dimensional TQFT [RT90, RT91, Wit89, Wal91, FHLT10, Hai24, Hai25].

One can now ask if a symmetry group can be gauged. In this paper, and in [VD23], the symmetry in question is the action of the orthogonal group $O(A)$ on Reshetikhin-Turaev theory (defined by the MTC $\mathbf{Vect}[A]$) acting via braided autoequivalences. As is argued in [VD23], **the anomaly theory encoding the projectivity (or ‘t Hooft anomaly) of this symmetry can be obtained by coupling the Crane-Yetter theory itself to an $O(A)$ -background field**.

We are considering the particularly simple case where the group is polarized: $A = L \oplus L^\vee$. In this case the Reshetikhin-Turaev theory in question is in fact a Turaev-Viro theory sending the point to the fusion category $\mathbf{Vect}[L]$.

Said differently, the fact that $O(A)$ acts on $A = L \oplus L^\vee$ can be leveraged to describe the projectivity of its action on σ_{BL}^3 . The same phenomenon is happening in our description of the integral transforms herein: Even though generic orthogonal group elements don’t preserve L within A , we leverage the fact that the group tautologically acts on A itself to construct the correspondences which quantize to give the projective action of $O(A)$ on σ_{BL}^3 .

1.1. Relationship with other work.

²Here “twice-categorified” refers to the categorical level of the quantization: the theory σ_X^d assigns to a point an object of a d -category, which for $d = 1$ is the vector space of functions on $\pi_0 X$. The theory σ_{BL}^3 , which assigns the fusion category $\mathbf{Vect}[L]$, is twice-categorified relative to this.

1.1.1. *What is new.* Several of the ingredients used here are known. The canonical projective action of $O(L \oplus L^\vee)$ on σ_{BL}^3 for $|L|$ odd, and its description as a projective theory, are established in [VD23], using [ENO10, EG18, CGPW16]. The isomorphism between $O(L \oplus L^\vee)$ and the group of invertible $\mathbf{Vect}[L]$ -bimodule categories is due to [ENO10]; the Morita equivalence between $\mathbf{Vect}[L]$ and $\mathbf{Rep}(L)$ appears in [EGNO15, FT22], and the corresponding domain wall is studied in [FT22]. The relationship between actions of G on $\mathbf{Vect}[L]^\tau$ and extensions of τ to $L \rtimes G$ is discussed in [ENO10, §11].

What is new in this paper is the following:

- the framework of (projective) semiclassical actions on decorated π -finite spaces, and the identification of their quantizations with (anomalous) module structures (Section 3);
- the construction of the symmetries induced by automorphisms of the base as quantizations of explicit decorated correspondences, in terms of cochain data (γ, μ, α) (Section 4);
- the analogue κ_{ev} of the BKS kernel, the associated twice-categorified Fourier transform, the identification of this transform with the canonical projective action of T_σ (Theorems 5.1 and 5.2), and the composition law for these kernels (Proposition 5.3 and Corollary 5.1).

1.1.2. *Other work.* The same Fourier-type transform was studied in [FT22], where they use it to encode Kramers-Wannier duality, with a focus on the 2-dimensional CFTs living on the boundary of the 3-dimensional theory in question. In this paper we focus primarily on the 3-dimensional theory as the thing being acted *on*, rather than the thing acting on its boundary theories.

The symmetries of three-dimensional finite abelian gauge theories, and their description via Lagrangian subgroups, have been studied from several points of view. Boundary conditions and surface defects in abelian Chern-Simons and Dijkgraaf-Witten theories are described by Lagrangian subgroups, and invertible surface defects of the theory for L correspond to elements of $O(L \oplus L^\vee)$ [KS11, FSV13, FS16]. On the algebraic side, Brauer-Picard groups of pointed fusion categories are described via Lagrangian subgroups in [NR14], and the higher structure of the Picard groupoid of a braided fusion category is studied in [DN13]. The obstructions O_3 and O_4 appear in the physics literature as obstructions to symmetry fractionalization and to gauging [BBCW19, Tac20]. Iterated spans and their (twisted) linearizations, which underlie Hypothesis Q, are studied in [Hau18, Mor10, Har20, HL13], and orbifolds of topological field theories with a group action, which correspond to gauging the symmetries considered here, in [SW17]. Finally, the closest classical analogue of the kernels of Section 5 is the canonical Weil representation over a finite field, constructed via kernels in [GH09, GH11]; see Example 3.

One of the main mathematical roles played by TQFTs is providing a source of manifold invariants. Just as (framed, oriented, etc.) 3-dimensional TQFTs give rise to invariants of closed (framed, oriented, etc.) 3-manifolds, the data of the action of a group G on a 3-dimensional TQFT upgrades the theory: It now provides an invariant of 3-manifolds equipped with a G -bundle (sometimes with a possibly flat connection, depending on the nature of the action of the group). Viewed through this lens, there is a priori a possible anomaly when trying to couple a 3-dimensional TQFT/invariant to a flat G -bundle when G is acting via the full orthogonal group $O(L \oplus L^\vee)$, whereas there is no obstruction when G is just acting via $\text{Aut}(L)$. For L an elementary abelian p -group with p odd, the former anomaly also vanishes (Example 3).

Just as the mathematical theory of geometric quantization is tightly tied to the theory of symplectic geometry and topology, the quantization of metric groups, as studied in this paper, is closely tied to the geometry of finite orthogonal groups. In geometric quantization, the local model is the Hilbert space of L^2 -functions on half of the variables. To glue these

local models coherently over some global symplectic manifold, one has to trivialize a certain H^2 -obstruction which can be accessed via the universal obstruction class in the cohomology of the symplectic group. Similarly, in the context of quantizing metric groups, the “local model” is a 3-dimensional TQFT **Vect** $[L]$. If we are given some base topological space, say a manifold/parameter space, with a principal $O(L \oplus L^\vee)$ -bundle, the anomaly studied in this paper pulls back to this base to give the obstruction to gluing these local models coherently over the base parameter space.

As is discussed in [FT22], this domain wall between Dijkgraaf-Witten for L and $L^\vee$ is a form of electromagnetic duality for finite 3-dimensional gauge theories. This domain wall, along with the action of the rest of the group, is encoded by an action of a 4-dimensional theory (or really a “quiche” [FMT22]) given by pure topological 4-dimensional G gauge theory. These bulk 4-dimensional theories also sit in various dualities. These theories are finite versions of the topological twists of SUSY Yang-Mills theory studied in the geometric Langlands program. The coupling of the 3-dimensional theory to a background G -bundle, or equivalently to the bulk 4-dimensional G gauge theory, is in turn analogous to the *relative* Langlands program, and the anomalies encountered herein are analogous to the degree 4 obstructions appearing in the relative Langlands program [BZSV23, §5.1], [BDF⁺25, §B], [Tel23].

1.1.3. Three-functor formalisms. The existence of the theories and morphisms between them studied in this paper follows from the cobordism hypothesis and [Hypothesis Q](#). The latter assumes the existence of a functor from a category of spans of π -finite groupoids to a prescribed symmetric monoidal ∞ -category $\mathcal{T}$. We interpret this as a “quantization” of π -finite groupoids. The study of theories arising in this way was introduced in [Kon88] and studied further in [Qui95, Tur10]. We will consider the fully local case introduced in [Fre94] and studied in [FHLT10, §3.8] and [FMT22, §A].

In a different context, three-functor formalisms are functors from a category of spans of objects of a fixed category (usually of algebro-geometric spaces such as varieties or stacks) to the category of ∞ -categories [Sch25]. Usually the correspondences are allowed to be decorated by morphisms to some coefficient category. [Hypothesis Q](#) is analogous: it asserts the existence of a symmetric monoidal functor out of a higher category of iterated (decorated) spans of π -finite groupoids, satisfying certain conditions. The constructions of this paper take place at the level of spans, and so make sense for any such functor. The theorems in dimension three, however, use the explicit description of Sum_3 valued in **Fus** ([Hypothesis Q'](#)) and the results of [ENO10].

1.2. Outline. In the rest of the introduction we provide a more detailed summary of the main results of this paper.

[Section 2](#) is an overview of topological symmetry, mostly following [FHLT10, FMT22]. Preliminaries regarding the target category and boundary/relative theories are discussed in [Section 2.1](#). In [Section 2.3](#) we discuss quantization of finite groupoids, and introduce [Hypothesis Q](#). In [Section 2.4](#) we review the notion of a module structure on a QFT from [FMT22].

[Section 3](#) consists of a discussion of semiclassical symmetries. Namely, we define linear and projective semiclassical actions in [Section 3.1](#), and discuss the anomaly/obstruction to extending a semiclassical action on a π -finite space X to a semiclassical action on the decorated space (X, τ) for some cocycle τ .

In [Section 4](#), we show that automorphisms of a finite group L preserving the class of a 3-cocycle τ act on the theory $\sigma_{BL, \tau}^3$ once the obstructions $d_2[\tau]$ and $d_3[\tau]$ in the Lyndon-Hochschild-Serre spectral sequence vanish, and canonically when τ itself is preserved. In [Section 4.1](#) we define the action up to isomorphism f , in [Section 4.2](#) we define a system of products m , and in [Section 4.3](#) we define an associator. In [Section 4.2.1](#) we discuss the obstruction $O_3(f)$,

and in [Section 4.3.1](#) we discuss the obstruction $O_4(f, m)$. In [Section 4.3.2](#) we state and prove [Theorem 4.1](#). The connection with semiclassical actions is discussed in [Section 4.4](#).

In [Section 5](#) we instead consider automorphisms of the full group $L \oplus L^\vee$ which preserve the quadratic form ev . In [Section 5.1](#) we define an analogue of the BKS kernel. The quantization of this, as a domain wall between TQFTs, is considered in [Section 5.2](#). In [Section 5.3](#), the local data defining the domain wall (a bimodule category) is discussed. This is related to the constructions in [\[ENO10\]](#) in [Section 5.4](#), to give a projective action of the full finite orthogonal group in [Section 5.5](#). In [Section 5.6](#) we compute the composition of these kernels (a form of Fourier inversion), show that together with the correspondences of [Section 4](#) they account for every element g with $gL^\vee \cap L^\vee = 0$ (and, for $L \simeq \mathbb{F}_p^n$, for all of $O(L \oplus L^\vee)$), and discuss the resulting anomaly, including the example $L = \mathbb{Z}/p$. [Section A](#) is dedicated to a discussion of some technical points regarding the Schur isomorphism, which are used in [Section 5](#).

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1.4. Orthogonal symmetries. Symmetries of the phase space which do not preserve a chosen Lagrangian act on the associated geometric quantization in a subtle way. One way of accessing the action is via the theory of integral transforms associated to BKS kernels. In [Section 5](#) we develop an analogue of this theory in the following context: The phase space is a polarized metric group $L \oplus L^\vee$ equipped with evaluation, and the symmetries are the elements of $O(L \oplus L^\vee)$.

In particular, for any two Lagrangians L and K in a metric group A , we consider the correspondence diagram:

$$\begin{array}{ccc} & A/(L \cap K) & \\ \swarrow \scriptstyle / (K/L \cap K) & & \searrow \scriptstyle / (L/L \cap K) \\ A/K & & A/L \end{array}$$

We make use of this diagram when K is the image of L under some $g \in O(A)$, and L and gL are transverse. This then becomes the correspondence:

$$\begin{array}{ccc} & A & \\ \swarrow & & \searrow \\ L & & gL \end{array}$$

This diagram defines a 1-morphism in the category of π -finite groupoids with 1-morphisms given by correspondences of such, 2-morphisms given by correspondences of correspondences, and so on. In [Section 2.3](#) we consider the particular flavor of the correspondence category where the groupoids can be decorated by $\mathbf{k}^\times$ -cocycles (where $\mathbf{k}$ is a fixed algebraically closed field of characteristic zero). Cohomologically distinguishable 2-cocycles on A define different 1-morphisms in this category.

[Section 2.3](#) discusses a very general construction of TQFTs from π -finite groupoids depending on [Hypothesis Q](#), which assumes that there is an appropriate “quantization” functor. This functor sends π -finite groupoids to objects of the chosen target category, and correspondences

to morphisms. For target category **Fus**, the Morita 3-category of $\mathbf{k}$ -linear fusion categories, this functor sends the groupoid BL to the fusion category $\mathbf{Vect}[L]$, and correspondences get sent to bimodule categories. Precomposing this functor with the mapping space functor $\text{Map}(-, BL)$ from $\mathbf{Bord}_3$ to the correspondence category, we obtain the TQFT σ_{BL}^3 , untwisted Dijkgraaf-Witten theory for the group L . Correspondingly, decorated correspondences define domain walls between the associated TQFTs.

When $A \simeq L \oplus L^\vee$ there is a canonical class in $H^2(A, \mathbf{k}^\times)$. Write κ_{ev} for any 2-cocycle representative of this class. The 1-morphism in the correspondence category

$$\begin{array}{ccc} & (A, \kappa_{\text{ev}}) & \\ \swarrow & & \searrow \\ L & & gL \end{array}$$

induces the integral transform from the 3-dimensional TQFT associated to L , σ_L^3 , and the one associated to gL , σ_{gL}^3 . I.e. κ_{ev} is the higher kernel of the integral transform defining the domain wall between these theories.

If g is an orthogonal homomorphism which sends L to $L^\vee$, then the associated domain wall can be thought of as a higher Fourier transform, as defined in [Definition 5.1](#):

$$\text{FT}: \sigma_{BL}^3 \rightarrow \sigma_{BL^\vee}^3$$

The assignment to the point is studied in [Section 5.3](#). As a $\mathbf{Vect}[L]$ - $\mathbf{Rep}(L)$ -bimodule category this is a Morita equivalence, as studied in [EGNO15, FT22], which only has a single simple object, i.e. is **Vect**. Note that this exchanges convolution on $\mathbf{Vect}[L]$ with the symmetric-monoidal structure on $\mathbf{Rep}(L)$, as the usual Fourier transform does.

After making a noncanonical choice, this is written as a $\mathbf{Vect}[L]$ - $\mathbf{Vect}[L]$ -bimodule in [Proposition 5.2](#), which again only has a single simple object and is of order 2.

In [Section 5.4](#) we establish a projective action of the full orthogonal group $O(L \oplus L^\vee)$ on σ_{BL}^3 by explicitly defining a 1-morphism in the correspondence category for every element $g \in O(L \oplus L^\vee)$. This uses the classification of bimodule categories over pointed fusion categories in [ENO10]. These 1-morphisms induce domain walls between the different models for the same 3-dimensional TQFT.

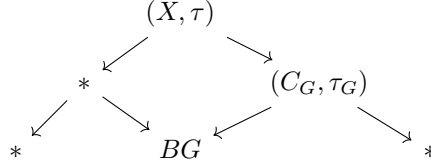
There is a certain element of $O(L \oplus L^\vee)$ of order 2, which sends L to $L^\vee$. The construction of a bimodule category from [ENO10] from this element is shown to agree with the assignment to the point of the twice-categorified Fourier transform ([Definition 5.1](#)) in [Theorem 5.1](#). This is then cast in field-theoretic terms in [Theorem 5.2](#):

Theorem. Assume [Hypothesis Q](#) and that L has odd order. The canonical projective action of $T_\sigma \in O(L \oplus L^\vee)$ on σ_{BL}^3 is via the domain wall FT ([Definition 5.1](#)).

1.5. Automorphisms of the base. Now we will consider an action of a finite group G on L preserving the class of τ , which, subject to the obstructions discussed below (and canonically when τ itself is preserved), defines an action of G on $\sigma_{BL, \tau}^3$. But first we will make some general considerations regarding actions of groups on semiclassical data.

1.5.1. Semiclassical action. Following ideas in [FMT22], we define a *semiclassical action* ([Definition 3.1](#)) of a finite group G on semiclassical data (X, τ) (where τ represents a class in $H^d(X, \mathbf{k}^\times)$) to be a correspondence (i.e. 1-morphism) between the trivial π -finite space to BG

such that when we pull back along $* \rightarrow BG$ we obtain (X, τ) :



I.e. this consists of an X -bundle over BG and an extension of τ to the total space C_G .

The semiclassical action is *projective* if BG is replaced by (BG, c) for some $(d+1)$ -cocycle on BG , and a trivialization defines a correspondence (1-morphism) from (BG, c) to BG (Definition 3.2). Composing the projective action with this correspondence “linearizes” the projective action to an ordinary one.

Assuming we have a semiclassical action of G on X , we would like to identify the obstruction to extending this to a semiclassical action of G on (X, τ) . I.e. we have an X -bundle over BG with total space C_G , and we seek the obstruction to extending τ from X to C_G .

The Serre spectral sequence provides a general method for answering this question. In Theorem 3.1, under some assumptions, we show that the spectral sequence obstruction encodes an anomaly/anomalous symmetry of the theory $\sigma_{X, \tau}^d$. Theorem 3.1 is a fully extended analogue, conditional on Hypothesis Q, of the descriptions of ‘t Hooft anomalies via spectral sequences in [Mü20, MS20, KT14]. The underlying algebraic input is the classical description of transgression in the Serre spectral sequence.

In Corollary 3.1, we note that the anomaly/projectivity agrees with (a restriction of) the top k -invariant of the higher automorphism groupoid. This is an analogue in arbitrary dimension of part of the arguments made in [ENO10, §A].

1.5.2. Three dimensions. Now we specialize to three dimensions and consider the particular space $X = BL$ for L a finite group with τ a group cocycle representing a class in $H^3(BL, \mathbf{k}^\times) \simeq H_{\text{Grp}}^3(L, \mathbf{k}^\times)$. Let G be a finite group, and consider a group homomorphism

$$\varphi: G \rightarrow \text{Aut}(L, \tau)$$

where $\text{Aut}(L, \tau)$ denotes the group of group automorphisms of L which preserve the cohomology class of τ .

The fully-extended TQFT

$$\sigma_{L, \tau}: \mathbf{Bord}_3^{\text{fr}} \rightarrow \mathbf{Fus}$$

(see Section 2) sends the point to

$$\mathcal{C} = \mathbf{Vect}[L]^\tau,$$

the fusion category of vector spaces graded by L , with convolution product twisted by τ .

Therefore, one avatar of a group acting on a category might be a map:

$$\mathbf{f}: G \rightarrow \text{Aut}_{\mathbf{Fus}}(\mathcal{C}).$$

In Section 4, we define such a map by explicitly defining the map on the underlying ordinary groups, and identifying the obstructions of [ENO10] for these data with differentials in the Lyndon-Hochschild-Serre spectral sequence. Any functor

$$BG \rightarrow B\text{Aut}_{\mathbf{Fus}}(\sigma_{BL, \tau}^3(*))$$

is equivalent to a (σ_{BG}^4, ρ) -module structure on $\sigma_{BL, \tau}^3$ by Hypothesis Q and the cobordism hypothesis. (See Proposition 2.2.) This shows the following, which is stated as Theorem 4.1 in the body of this paper.

Theorem. Let L and G be finite groups, let $\tau \in Z_{\text{Grp}}^3(L, \mathbf{k}^\times)$, and let $\varphi: G \rightarrow \text{Aut}(L, \tau)$ be a homomorphism. A triple (γ, μ, α) satisfying the cocycle conditions of [Section 4](#) (equivalently, an extension of τ to a 3-cocycle on $L \rtimes_\varphi G$) defines a map of 3-groups $G \rightarrow \text{Aut}_{\mathbf{Fus}}(\mathbf{Vect}[L]^\tau)$, and hence, assuming [Hypothesis Q](#), a (σ_{BG}^4, ρ) -module structure on $\sigma_{BL, \tau}^3$. Such a triple exists if and only if $d_2[\tau]$ and $d_3[\tau]$ vanish in the Lyndon-Hochschild-Serre spectral sequence. If φ preserves τ on the nose (e.g. if τ is trivial), there is a canonical such triple.

Such a triple is the same thing as an extension of τ to a group 3-cocycle τ_G on $L \rtimes_\varphi G$ ([Proposition 4.2](#)). The quantization of the correspondence

$$\begin{array}{ccc} & (C_G, \tau_G) & \\ \swarrow & & \searrow \\ BG & & * \end{array}$$

defines a (σ_{BG}^4, ρ) -module structure on $\sigma_{BL, \tau}^3$. This module structure equivalently defines a functor

$$BG \rightarrow B \text{Aut}_{\mathbf{Fus}}(\mathbf{Vect}[L]^\tau) ,$$

and we expect that it recovers the map (f, m, a) . This is discussed in more detail in [Section 4.4](#).

2. TOPOLOGICAL FIELD THEORY

Let $d \in \mathbb{Z}^{\geq 0}$. Given a symmetric-monoidal $(\infty, d+1)$ -category $\mathcal{T}$, which will be the target for our theories, write $\mathcal{T}^\sim$ for the groupoid obtained by discarding all non-invertible morphisms at all levels. We will assume $\mathcal{T}$ has duals.³ We will also assume that $\Omega^{d+1}\mathcal{T} = \mathbf{k}$ for $\mathbf{k}$ an algebraically closed field of characteristic zero. Throughout the paper, we will write $\mathbf{Vect}$ for the $\mathbf{k}$ -linear category of finite-dimensional vector spaces over $\mathbf{k}$.

Remark 1. Besides having duals and satisfying $\Omega^{d+1}\mathcal{T} = \mathbf{k}$, we will occasionally assume that $\mathcal{T}$ satisfies an additional hypothesis. In particular, when we discuss module structures in the sense of [\[FMT22\]](#), we will assume that the target $\mathcal{T}$ is sufficiently additive to support quantization of π -finite spaces in the sense of [\[FHLT10, §8\]](#) and [\[FMT22, §A\]](#) (this is [Hypothesis Q](#)).

2.1. Relative/boundary theories. A symmetric monoidal functor

$$\alpha: \mathbf{Bord}_d^X \rightarrow \mathcal{T}$$

is a *once-categorified d -dimensional X -TQFT*, where $\mathcal{T}$ is the fixed target from the beginning of [Section 2](#). Let

$$1_X: \mathbf{Bord}_d^X \rightarrow \mathcal{T}$$

denote the trivial once-categorified d -dimensional X -TQFT. Recall the notion of a relative theory [\[FT14\]](#). These are also called *twisted theories* [\[ST11, JFS17\]](#).

Definition. A theory defined right-relative to α is a lax natural transformation (in the sense of [\[JFS17\]](#))

$$\alpha \rightarrow 1 .$$

A theory defined left-relative to α is a lax natural transformation

$$1 \rightarrow \alpha .$$

³Otherwise replace $\mathcal{T}$ with the subcategory consisting of the fully-dualizable objects of $\mathcal{T}$.

Remark 2. Recall the definition of lax (resp. oplax) natural transformations from [JFS17]. Consider the lax and oplax arrow categories $\mathcal{T}^\downarrow$ and $\mathcal{T}^\rightarrow$ of [JFS17], and the source and target functors $s, t: \mathcal{T}^* \rightarrow \mathcal{T}$ for $* = \downarrow, \rightarrow$. Following [JFS17], a lax (resp. oplax) natural transformation $\alpha \rightarrow 1$ is a functor

$$(2.1) \quad F_\alpha: \mathbf{Bord}_d^X \rightarrow \mathcal{T}^\downarrow \quad \left(\text{resp. } F_\alpha: \mathbf{Bord}_d^X \rightarrow \mathcal{T}^\rightarrow \right)$$

such that $s \circ F_\alpha = \alpha$, and $t \circ F_\alpha = 1$.

Throughout, we will use the lax version, as written in the above definition of relative theories. The reason we use the lax version is that, as noted in [JFS17, Example 7.3], the *lax* natural transformations from the trivial theory to itself consist of theories of dimension lower ([JFS17, Theorem 7.4]) whereas the same is not true when lax is replaced with oplax. We need the analogous result for X -theories in order to establish a trivialized anomalous theory $1 \xrightarrow{\sim} \alpha \rightarrow 1$ as an honest theory of one dimension lower.

Also noted in [JFS17, Example 7.3], is the fact that oplax natural transformations are “elements” $F_\sigma(M): 1 \rightarrow \alpha(M)$ for M a *closed* bordism of any codimension, which is for example the point of view taken in [FT14]. Oplax natural transformations are also used in [FT21]. We expect all results to hold upon replacing all lax natural transformations with oplax ones (compare [VD23]).

Remark 3. Often a once-categorified d -dimensional theory α extends to a $(d+1)$ -dimensional theory:

$$\begin{array}{ccc} & & \mathcal{T} \\ & \nearrow \alpha & \uparrow \\ \mathbf{Bord}_d^X & \hookrightarrow & \mathbf{Bord}_{d+1}^X \end{array}$$

In this case, a relative theory $\alpha \rightarrow 1$ can be upgraded to what is called a *boundary theory*. Boundary theories are defined as functors out of the extended bordism category $\mathbf{Bord}_{d+1}^{X,\partial}$, described in [Lur09, §4.3]. The connection with the notion defined here is made in [JFS17, Theorem 7.15].

2.2. Projective/anomalous theories. Recall the projectivization $\mathbb{P}\Omega\mathcal{T}$ of the target category as defined in [VD23]. Explicitly, $\mathbb{P}\Omega\mathcal{T} := \mathcal{T}[0] \times_{\mathcal{T}} \mathcal{T}^\downarrow \times_{\mathcal{T}} \mathcal{T}^\times$ consists of arrows in $\mathcal{T}$ between an invertible object of $\mathcal{T}$ and the unit. For instance, for $\mathcal{T} = \mathbf{Alg}$ an object is a module over an Azumaya algebra, which up to Morita equivalence is a vector space defined only up to scalar.

Definition 2.1. A d -dimensional *projective* (X, ζ) -*TQFT* is a symmetric-monoidal functor:

$$\overline{F}: \mathbf{Bord}_d^{(X, \zeta)} \rightarrow \mathbb{P}\Omega\mathcal{T}.$$

Recall by construction of $\mathbb{P}\Omega\mathcal{T}$ there is a canonical ‘source functor’ (forgetting the module) $s: \mathbb{P}\Omega\mathcal{T} \rightarrow \mathcal{T}^\times$. Given such a theory $\overline{F}$, composing with the functor s results in an invertible theory:

$$\alpha: \mathbf{Bord}_d^{(X, \zeta)} \xrightarrow{\overline{F}} \mathbb{P}\Omega\mathcal{T} \xrightarrow{s} \mathcal{T}^\times.$$

This theory is the *projectivity* of the projective theory $\overline{F}$. Sometimes to emphasize that a theory is *not* projective, we will call it *linear*.

Passing to the assignment to the point, we obtain the space of projective families of objects of $\Omega\mathcal{T}$ over X , $\text{Hom}(X, \mathbb{P}\Omega\mathcal{T})$. The target functor $t: \mathbb{P}\Omega\mathcal{T}$ induces a map to the space of families of invertible objects of $\mathcal{T}$ over X :

$$\text{Hom}(X, \mathbb{P}\Omega\mathcal{T}) \rightarrow \text{Hom}(X, \mathcal{T}^\times)$$

The fiber over any particular projectivity $\tau: X \rightarrow \mathcal{T}^\times$, which we will write as $\text{Hom}(X, \mathbb{P}\Omega\mathcal{T})_\tau$, consists of projective families of objects of $\Omega\mathcal{T}$ with projectivity τ .

2.3. TQFTs associated to π -finite spaces. Let X be a space (i.e. higher groupoid) which is (connected, pointed, and) π -finite⁴. There is a recipe for constructing a TQFT using X , which was introduced in [Kon88] and studied further in [Qui95, Tur10]. We will consider the fully local case introduced in [Fre94] and studied in [FHLT10, §3,8] and [FMT22, §A]. In the 1-dimensional case, the finite path integral has mathematically rigorous foundations in the ∞ -categorical setting [Har20]. Related work in a higher setting is the subject of an upcoming work of Claudia Scheimbauer and Tashi Walde.

2.3.1. The quantization map. We will proceed heuristically, following [FHLT10, §3] and [FMT22, §A.2], to fix notation and describe expectations which will eventually be stated and assumed in [Hypothesis Q](#).

Let $\mathbf{Fam}_{d+1}$ denote the category with objects π -finite spaces, 1-morphisms given by correspondences of π -finite spaces, 2-morphisms given by correspondences of correspondences, and so on until level $(d+1)$. (Two $(d+1)$ -morphisms are regarded as identical if they are equivalent.)

Let $\mathcal{T}$ be the arbitrary symmetric monoidal target with duals, fixed at the beginning of [Section 2](#). Let Y be an object of $\mathbf{Fam}_{d+1}$. A local system on Y valued in $\mathcal{T}$ is a functor $Y \rightarrow \mathcal{T}$. Write $\mathbf{Fam}_{d+1}(\mathcal{T})$ for the category of π -finite spaces equipped with a local system valued in $\mathcal{T}$, with morphisms given by correspondences equipped with morphisms between the pulled-back local systems.

For example, $\mathbf{Fam}_{d+1}(B^{d+1}\mathbf{k}^\times)$ has objects given by pairs (Y, τ) , where τ is a cocycle

$$\tau: Y \rightarrow B^{d+1}\mathbf{k}^\times$$

representing a class in $H^{d+1}(Y, \mathbf{k}^\times)$. A morphism is a correspondence:

$$(2.2) \quad \begin{array}{ccc} & (E, \mu) & \\ \swarrow p_1 & & \searrow p_2 \\ (Y_1, \tau_1) & & (Y_2, \tau_2) \end{array}$$

where $\mu: E \rightarrow B^d\mathbf{k}^\times$ satisfies

$$(2.3) \quad d\mu = (p_1^*\tau_1)^{-1} \cdot (p_2^*\tau_2) \ .$$

2-morphisms are correspondences between correspondences with a similar condition on the cocycles, and so on to define morphisms up to level $(d+1)$.

Recall we assumed $\Omega^{d+1}\mathcal{T} = \mathbf{k}$. Therefore there is a natural functor: $B^{d+1}\mathbf{k}^\times \rightarrow \mathcal{T}$ inducing a functor:

$$(2.4) \quad \mathbf{Fam}_{d+1}(B^{d+1}\mathbf{k}^\times) \rightarrow \mathbf{Fam}_{d+1}(\mathcal{T}) \ .$$

We will assume that the following holds.

Hypothesis Q. For $\mathcal{T}$ appropriately additive⁵, there is a quantization⁶ functor

$$(2.5) \quad \text{Sum}_{d+1}: \mathbf{Fam}_{d+1}(B^{d+1}\mathbf{k}^\times) \rightarrow \mathcal{T}$$

⁴This means X has finitely many homotopy groups, each of which is finite.

⁵As is explained in [FHLT10, §3], for X an ordinary groupoid, then $\mathcal{T}$ must be additive in the sense that the colimit $\varinjlim_{x \in X} \tau(x)$ in $\mathcal{T}$ exists, and agrees with the limit $\varprojlim_{x \in X} \tau(x)$, where we are regarding τ as defining a $\mathcal{T}$ -valued local system on X by (2.4).

⁶See [FMT22, Remark A.7.1] where it is explained how $\text{Sum}_{d+1}(X)$ (and therefore σ_X^{d+1}) can be obtained by integrating over fluctuating fields.

such that there is an invertible natural transformation between

$$(X, \tau) \mapsto \mathrm{Hom}_{\mathcal{T}}(1, \mathrm{Sum}_{d+1}(X, \tau))$$

and

$$(X, \tau) \mapsto \mathrm{Hom}_{\tau}(\mathrm{hofib}(\tau), \Omega\mathcal{T})$$

viewed as functors out of $\mathbf{Fam}_{d+1}(B^{d+1}\mathbf{k}^{\times})$.

Here $\mathrm{Hom}_{\tau}(\mathrm{hofib}(\tau), \Omega\mathcal{T})$ denotes the functors $\mathrm{hofib}(\tau) \rightarrow \Omega\mathcal{T}^{\sim}$ whose restriction to each fiber $B^d\mathbf{k}^{\times}$ is the canonical inclusion of scalars, and $\mathrm{Hom}_{\mathcal{T}}(1, -)$ denotes the $(d+1)$ -times left-adjunctible 1-morphisms (as in [JFS17, Corollary 7.7]). The second assignment is made functorial in correspondences by pull-push: a correspondence $X_1 \xleftarrow{p_1} E \xrightarrow{p_2} X_2$ as in (2.2) acts by pulling back along p_1 , transporting along the equivalence $\mathrm{hofib}(p_1^*\tau_1) \simeq \mathrm{hofib}(p_2^*\tau_2)$ over E induced by μ , and pushing forward along p_2 . The push-forward is the colimit over the fibers of p_2 , which exists and agrees with the corresponding limit by the additivity assumption on $\mathcal{T}$ [Har20].

Remark 4. Unpacking the existence of the natural transformation in [Hypothesis Q](#), we see that it ensures that we have equivalences

$$(2.6) \quad \mathrm{Hom}_{\mathcal{T}}(1, \mathrm{Sum}_{d+1}(X, \tau)) \simeq \mathrm{Hom}_{\tau}(\mathrm{hofib}(\tau), \Omega\mathcal{T})$$

for all objects (X, τ) of $\mathbf{Fam}_{d+1}(B^{d+1}\mathbf{k}^{\times})$, and given a morphism from (X_1, τ_1) to (X_2, τ_2) in $\mathbf{Fam}_{d+1}(B^{d+1}\mathbf{k}^{\times})$, we have a commuting diagram:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{T}}(1, \mathrm{Sum}_{d+1}(X_1, \tau_1)) & \simeq & \mathrm{Hom}_{\tau_1}(\mathrm{hofib}(\tau_1), \Omega\mathcal{T}) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathcal{T}}(1, \mathrm{Sum}_{d+1}(X_2, \tau_2)) & \simeq & \mathrm{Hom}_{\tau_2}(\mathrm{hofib}(\tau_2), \Omega\mathcal{T}) \end{array}$$

I.e. τ -twisted higher representations of ΩX are equivalent to morphisms from 1 to $\mathrm{Sum}_{d+1}(X, \tau)$ in $\mathcal{T}$. This generalizes the classical fact about modules over the group algebra and G -representations, as in [Example 1 \(ii\)](#).

Remark 5. In [FHLT10, §8.2], the map Sum_n is constructed at the level of objects and shown to be a functor up to 2-morphisms. In particular, the cases in [Example 1](#) are worked out in [FHLT10, §8.1, 8.3].

Remark 6. Correspondences $* \leftarrow E \rightarrow *$ in $\mathbf{Fam}_{d+1}(B^{d+1}\mathbf{k}^{\times})$ are spaces E decorated by d -cocycles, i.e. $\mathrm{End}_{\mathbf{Fam}_{d+1}(B^{d+1}\mathbf{k}^{\times})}(*) \simeq \mathbf{Fam}_d(B^d\mathbf{k}^{\times})$. Therefore [Hypothesis Q](#) for Sum_{d+1} induces

$$\mathrm{Sum}_d := \Omega_* \mathrm{Sum}_{d+1} : \mathbf{Fam}_d(B^d\mathbf{k}^{\times}) \rightarrow \Omega\mathcal{T},$$

and this is the functor we use for d -dimensional theories $\sigma_{X, \tau}^d$. In particular, no separate hypothesis is needed in each dimension.

Example 1. (i) Let $\mathcal{T} = \mathbf{Vect}$. Then $\mathrm{Sum}_1(X) = \mathbf{k}(\pi_0 X)$ is the vector space of $\mathbf{k}$ -valued functions on $\pi_0 X$. [Hypothesis Q](#) is satisfied, since the natural transformation (2.6) required in [Hypothesis Q](#) is:

$$\mathrm{Hom}_{\mathbf{Vect}}(1, \mathrm{Sum}_1(X)) \simeq \mathrm{Hom}(X, \mathbf{k}) \simeq \mathrm{Hom}(\pi_0 X, \mathbf{k}).$$

(ii) Set $X = BG$ for a finite group G , and let $\mathcal{T} = \mathbf{Alg}$ be the Morita 2-category of algebras. Let $\mathrm{Sum}_2(BG) = \mathbf{k}[G]$ be the group algebra. [Hypothesis Q](#) is satisfied, since the natural transformation (2.6) required in [Hypothesis Q](#) is:

$$\mathrm{Hom}_{\mathbf{Alg}}(1, \mathrm{Sum}_2(BG)) = \mathbf{k}[G]\text{-mod} \simeq \mathbf{Rep}(G) \simeq \mathrm{Hom}(BG, \mathbf{Vect}).$$

We can equip BG with a 2-cocycle μ (i.e. a group 2-cocycle) which twists the convolution structure on the group algebra, resulting in $\text{Sum}_2(BG, \mu)$.

- (iii) Set $X = BG$. Let $d = 2$, and take $\mathcal{T}$ to be the Morita 3-category of fusion categories. Then $\text{Sum}_3(BG)$ is $\mathbf{Vect}[G]$, the fusion category of vector bundles on G with convolution. We can equip BG with a 3-cocycle α which twists the fusion structure on $\mathbf{Vect}[G]$, yielding $\sigma_{BG, \alpha}^3(*) = \mathbf{Vect}[G]^\alpha$. This is [FMT22, Example A.65].
- (iv) Continuing the previous example, Sum_3 can be extended to certain 1-morphisms in $\mathbf{Fam}_3(B^3\mathbf{k}^\times)$ as follows. Consider a 1-morphism in $\mathbf{Fam}_3(B^3\mathbf{k}^\times)$ of the form

$$\begin{array}{ccc} & (BC, \kappa) & \\ \swarrow & & \searrow \\ BG & & BH \end{array}$$

where $C \subset G \oplus H$ is a subgroup, and κ is a 2-cocycle on C . The alternating bicharacter $\text{alt } \kappa$ on C (see Section A) classifies a $\mathbf{Vect}[G]$ - $\mathbf{Vect}[H]$ -bimodule category, as explained in [ENO10, §2.7]. Under Hypothesis Q' below, Sum_3 sends the 1-morphism (BC, κ) to this bimodule category.

- E.g. if $C = G \times H$ is the whole group then, for any bicharacter, the corresponding bimodule has a single simple object.
- E.g. if $G = H$ and C is the graph of a group automorphism of G , the bimodule category is the identity bimodule $\mathbf{Vect}[G]$ with action twisted by the group automorphism.
- (v) Set $X = B^2G$ (where now G is necessarily abelian). Let $d = 3$ and take $\mathcal{T}$ to be the Morita 4-category of braided monoidal categories, $\mathbf{BrFus}$ (see [BJS21, §3]). Consider a cocycle $\tau: B^2G \rightarrow B^4\mathbf{k}^\times$. It is a classical theorem of Eilenberg-Mac Lane [EM46] that cohomology classes in $H^4(B^2G, \mathbf{k}^\times)$ correspond to quadratic forms $q_\tau: G \rightarrow \mathbf{k}^\times$, and equivalently to classes of abelian 3-cocycles on G . Then $\text{Sum}_4(B^2G, \tau)$ is $\mathbf{Vect}[G]$ with convolution, and with associator and braiding given by an abelian 3-cocycle (τ_0, b) representing $[\tau]$. In particular the self-braiding of $\mathbf{k}_g$ is $b(g, g) = q_\tau(g)$.
- (vi) Let $X = BG$ and $d = 3$, only now we would like to use the category of monoidal 2-categories. Then $\text{Sum}_4(BG)$ is the fusion 2-category (in the sense of [DR18]) of G -graded 2-vector spaces [DR18, Construction 2.1.13]. The dualizability of fusion 2-categories is the subject of [Dé24, Theorem C].

The description of Sum_3 on 1-morphisms in Example 1 Item (iv) does not follow from Hypothesis Q, which only constrains $\text{Hom}_{\mathcal{T}}(1, -)$. When we use it, we will assume it explicitly:

Hypothesis Q'. For $\mathcal{T} = \mathbf{Fus}$, the functor Sum_3 of Hypothesis Q sends BG to $\mathbf{Vect}[G]$, and sends a 1-morphism $BG \leftarrow (BC, \kappa) \rightarrow BH$, with $C \subset G \times H$, to the $\mathbf{Vect}[G]$ - $\mathbf{Vect}[H]$ -bimodule category associated to $(C, \text{alt } \kappa)$ in [ENO10, §2.7].

2.3.2. TQFTs from the quantization map. The upshot of assuming Hypothesis Q is that, for a fixed object (X, τ) of $\mathbf{Fam}_{d+1}(B^{d+1}\mathbf{k}^\times)$, (2.5) can be composed with the mapping space functor⁷ to obtain the theory

$$\sigma_{X, \tau}^{d+1}: \mathbf{Bord}_{d+1}^{\text{fr}} \xrightarrow{\pi_{\leq d+1} \text{Map}(-, X)} \mathbf{Fam}_{d+1}(\mathcal{T}) \xrightarrow{\text{Sum}_{d+1}} \mathcal{T}.$$

Remark 7. As is remarked in [FMT22, §A] and [FHLT10, §3,8], $\sigma_{X, \tau}^{d+1}$ can be upgraded to an oriented theory, and if τ is trivial then it can even be upgraded to an unoriented theory. We will work with the underlying framed theories in this paper.

The following results follow directly from Hypothesis Q and the cobordism hypothesis.

⁷Note that (2.4) allows us to construct a $\mathcal{T}$ -valued local system from the cocycle τ .

Proposition 2.1. Assume *Hypothesis Q*. A morphism from (X_1, τ_1) to (X_2, τ_2) in

$$\mathbf{Fam}_{d+1}(B^{d+1}\mathbf{k}^\times)$$

(i.e. correspondence as in (2.2)) induces a bimodule (i.e. domain wall)

$$\sigma_{X_1, \tau_1}^{d+1} \rightarrow \sigma_{X_2, \tau_2}^{d+1}.$$

Proof. A correspondence as in (2.2) has left and right adjoints in $\mathbf{Fam}_{d+1}(B^{d+1}\mathbf{k}^\times)$, given by the reversed correspondence, and the same holds at every level. Therefore its image under Sum_{d+1} is a $(d+1)$ -times left-adjunctible 1-morphism $\text{Sum}_{d+1}(X_1, \tau_1) \rightarrow \text{Sum}_{d+1}(X_2, \tau_2)$ in $\mathcal{T}$. By [JFS17, Corollary 7.7], this classifies a domain wall $\sigma_{X_1, \tau_1}^{d+1} \rightarrow \sigma_{X_2, \tau_2}^{d+1}$. $\square$

Remark 8. Proposition 2.1 can be shown directly (i.e. without the cobordism hypothesis) since a correspondence of spaces $X \leftarrow C \rightarrow Y$ defines a correspondence of mapping spaces $\text{Map}(M, X) \leftarrow \text{Map}(M, C) \rightarrow \text{Map}(M, Y)$ for any bordism M . This is the perspective taken in [FMT22].

Proposition 2.2. Assume *Hypothesis Q*. Every boundary theory for $\sigma_{X, \tau}^{d+1}$ is classified by a symmetric-monoidal functor:

$$\mathbf{Bord}_d^{\text{hofib}(\tau)} \rightarrow \Omega\mathcal{T}$$

whose classifying map lies in $\text{Hom}_\tau(\text{hofib}(\tau), \Omega\mathcal{T})$. Equivalently [VD23, §3.3], it is classified by a projective theory $\mathbf{Bord}_d^X \rightarrow \mathbb{P}\Omega\mathcal{T}$ with projectivity τ (Section 2.2). If τ is trivial, then the boundary theories are classified by symmetric-monoidal functors:

$$\mathbf{Bord}_d^X \rightarrow \Omega\mathcal{T}.$$

Proof. By [JFS17, Theorem 7.15], the boundary theories for $\sigma_{X, \tau}^{d+1}$ are classified by the $(d+1)$ -times left-adjunctible 1-morphisms $1 \rightarrow \sigma_{X, \tau}^{d+1}(\ast) = \text{Sum}_{d+1}(X, \tau)$, i.e. by the left-hand side of (2.6). By the cobordism hypothesis for X -theories [Lur09], the symmetric monoidal functors from $\mathbf{Bord}_d^{\text{hofib}(\tau)}$ to $\Omega\mathcal{T}$ are classified by $\text{Hom}(\text{hofib}(\tau), \Omega\mathcal{T}^\sim)$, and the right-hand side of (2.6) is the subspace Hom_τ . If τ is trivial, then a trivialization identifies $\text{hofib}(\tau) \simeq X \times B^d\mathbf{k}^\times$, so the right-hand side of (2.6) is equivalent to functors from X to $\Omega\mathcal{T}^\sim$. $\square$

2.4. Module structures. We summarize the material used in our construction from [FMT22]. See [FMT22, §3] for a more detailed discussion of these definitions.

Let

$$\sigma: \mathbf{Bord}_{d+1}^{\text{fr}} \rightarrow \mathcal{T}$$

be a $(d+1)$ -dimensional TQFT valued in the fixed target $\mathcal{T}$ from the beginning of Section 2. Recall the following definition in [FMT22].

Definition. A d -dimensional quiche is a pair (σ, ρ) in which ρ is a right topological boundary theory (or *right σ -module*), which we will write as $\rho: \sigma \rightarrow 1$.

Remark 9. All of the quiches considered in this paper will be of the form $(\sigma_{X, \tau}, \rho_{X, \tau})$, where $\sigma_{X, \tau}$ is the theory associated to (X, τ) as in Section 2.3, and the (right) boundary theory $\rho_{X, \tau}$ is the natural transformation induced by the correspondence diagram:

$$\begin{array}{ccc} & \ast & \\ \swarrow & & \searrow \\ (X, \tau) & & \ast \end{array}$$

as in [Proposition 2.1](#). Here the correspondence uses a trivialization of the restriction of τ to the basepoint. Given a pointed space X , we will always write $\rho_{X,\tau}$ for this boundary theory.

A quiche is an abstract symmetry datum, in the same sense as an algebra. The following definition is the analogue of a module, i.e. a realization of the quiche as symmetries of a given theory.

Definition. Let (σ, ρ) be a d -dimensional quiche, and let F be a d -dimensional TQFT. A (σ, ρ) -module structure on F is the pair (F_σ, θ) where F_σ is a (left) boundary theory $F_\sigma: 1 \rightarrow \sigma$ which is equipped with an isomorphism of d -dimensional theories:

$$\theta: \rho \otimes_\sigma F_\sigma \xrightarrow{\sim} F.$$

3. SEMICLASSICAL SYMMETRY

3.1. Semiclassical group actions. Fix $d \in \mathbb{Z}^{\geq 0}$, and let X be a π -finite space with cocycle $\tau: X \rightarrow B^d \mathbf{k}^\times$.

Definition 3.1. A semiclassical action of a group G on (X, τ) is a decorated correspondence of spaces from $*$ to BG , i.e. a 1-morphism in $\mathbf{Fam}_{d+1}(B^{d+1} \mathbf{k}^\times)$, where $*$ and BG carry the trivial $(d+1)$ -cocycle, so that τ_G is a d -cocycle:

$$(3.1) \quad \begin{array}{ccc} & (C_G, \tau_G) & \\ \swarrow & & \searrow \pi \\ * & & BG \end{array}$$

such that composing with the pointing $BG \leftarrow * \rightarrow *$ (which induces ρ_{BG}) gives (X, τ) . That is, we have a pullback

$$\begin{array}{ccccc} & & (X, \tau) & & \\ & \swarrow & & \searrow & \\ & (C_G, \tau_G) & & * & \\ \swarrow & & \searrow & \swarrow & \searrow \\ * & & BG & & * \end{array}$$

together with an identification of the restriction of τ_G to X with τ .

I.e. a semiclassical action of G on (X, τ) is just an X -bundle over BG , and an extension of τ to the total-space of this bundle.

Example 2. Let $X = BL$ for a finite group L . For trivial τ , there is always a canonical semiclassical action of $\text{Aut}_{\mathbf{Grp}}(L)$ given by

$$C_{\text{Aut}} = B(L \rtimes \text{Aut}_{\mathbf{Grp}}(L))$$

since the following is a pullback diagram:

$$\begin{array}{ccccc} & & BL & & \\ & \swarrow & & \searrow & \\ & * & & B(L \rtimes \text{Aut}_{\mathbf{Grp}}(L)) & \\ \swarrow & & \searrow & \swarrow & \searrow \\ * & & B \text{Aut}_{\mathbf{Grp}}(L) & & * \end{array}$$

The same holds for any $\varphi: G \rightarrow \text{Aut}_{\mathbf{Grp}}(L)$. Namely,

$$C_G = B(L \rtimes_\varphi G)$$

is the pullback

$$\begin{array}{ccccc}
 & & C_G & & \\
 & \swarrow & & \searrow & \\
 BG & & & & C_{\text{Aut}} \\
 \swarrow & & \searrow^{B\varphi} & \swarrow & \searrow \\
 BG & & B \text{Aut}_{\mathbf{Grp}}(L) & & *
 \end{array}$$

and this defines a semiclassical action because we also have a pullback

$$\begin{array}{ccccc}
 & & X & & \\
 & \swarrow & & \searrow & \\
 & & * & & C_G \\
 \swarrow & & \searrow & \swarrow & \searrow \\
 * & & BG & & *
 \end{array}$$

Definition 3.2. A *projective* semiclassical action of a group G on (X, τ) with projectivity a (normalized) cocycle $c: BG \rightarrow B^{d+1}\mathbf{k}^\times$ is a 1-morphism in $\mathbf{Fam}_{d+1}(B^{d+1}\mathbf{k}^\times)$ from $*$ to (BG, c) :

$$\begin{array}{ccc}
 (C_G, \tau_G) & & \\
 \swarrow & \searrow^{\pi} & \\
 * & & (BG, c)
 \end{array}$$

That is, by (2.3), a d -cochain τ_G on C_G with $d\tau_G = \pi^*c$, such that composing with the pointing of (BG, c) gives (X, τ) , as in Definition 3.1.

3.2. Spectral sequence obstruction. Consider a semiclassical action C_G of G on X as in Definition 3.1 (i.e. the twist is trivial). Now, given some nontrivial τ , we can pose the question of whether there is a semiclassical action (C_G, τ_G) of G on (X, τ) . I.e. we are asking if the cocycle τ extends to a cocycle τ_G on C_G :

$$\begin{array}{ccc}
 & B^d\mathbf{k}^\times & \\
 \tau \nearrow & & \nwarrow \tau_G \\
 X & \xrightarrow{\quad} & C_G \\
 \downarrow & & \downarrow \\
 * & \xrightarrow{\quad} & BG
 \end{array}$$

The Serre spectral sequence provides a general method for answering this question. The first quadrant Serre spectral sequence corresponding to the fibration

$$\begin{array}{ccc}
 X & \xrightarrow{i} & C_G \\
 & & \downarrow \\
 & & BG
 \end{array}$$

is

$$E_2^{p,q} = H^p(BG, H^q(X, \mathbf{k}^\times)) \Rightarrow H^{p+q}(C_G, \mathbf{k}^\times) .$$

Recall the differentials:

$$d_r^{p,q}: E_r^{p,q} \rightarrow E_r^{p+r, q-r+1} .$$

The group G acts on $H^d(X, \mathbf{k}^\times)$ (induced by conjugation in ΩC_G) and every cocycle on C_G is invariant under this action, so we insist that

$$\tau \in H^d(X, \mathbf{k}^\times)^G .$$

Then we have that

$$\tau \in \text{im}(i^*) = E_{d+2}^{0,d} \hookrightarrow H^d(X, \mathbf{k}^\times)^G$$

if and only if

$$(3.2) \quad d_k^{0,d} \tau = d_k \tau = 0 \in E_k^{k,d+1-k}.$$

Here $2 \leq k \leq d+1$.

Theorem 3.1. *Assume [Hypothesis Q](#). Let C_G be a semiclassical action of G on X as above, and let τ represent a class in $H^d(X, \mathbf{k}^\times)^G$. Suppose the first $(d-1)$ obstructions (3.2) vanish, i.e. $d_k[\tau] = 0$ for $2 \leq k \leq d$.*

- (i) *There is a d -cochain τ_G on C_G restricting to τ on X , with $d\tau_G = \pi^*c$ for a normalized $(d+1)$ -cocycle c on BG . Any such class $[c]$ maps to $d_{d+1}[\tau]$ under $H^{d+1}(BG, \mathbf{k}^\times) \rightarrow E_{d+1}^{d+1,0}$, so $[c]$ is only determined modulo the images of $d_2, \dots, d_d$ in $H^{d+1}(BG, \mathbf{k}^\times)$.*
- (ii) *For any such choice, (C_G, τ_G) is a projective semiclassical action with projectivity c , and it quantizes to a $(\sigma_{BG,c}^{d+1}, \rho_{BG,c})$ -module structure on $\sigma_{X,\tau}^d$, i.e. an anomalous action of G on $\sigma_{X,\tau}^d$ with anomaly theory classified by c .*
- (iii) *The action can be chosen to be linear, i.e. τ extends to a cocycle on C_G and so defines a semiclassical action on (X, τ) , if and only if $d_{d+1}[\tau] = 0$.*

Proof. By a classical theorem of Serre, a class in $H^d(X, \mathbf{k}^\times)$ survives to $E_{d+1}^{0,d}$ if and only if it is transgressive, i.e. it is represented by the restriction of a cochain on C_G whose coboundary is pulled back from BG , and d_{d+1} is then the transgression [Ser51, McC01]. This gives (i), after replacing τ by a cohomologous cocycle (which does not change $\sigma_{X,\tau}^d$ up to equivalence); the indeterminacy of $[c]$ is the indeterminacy of the transgression. Part (iii) follows since $E_{d+2}^{0,d} = E_\infty^{0,d}$ is the image of the restriction $H^d(C_G, \mathbf{k}^\times) \rightarrow H^d(X, \mathbf{k}^\times)$.

For (ii), by (2.3) the condition $d\tau_G = \pi^*c$ says that (C_G, τ_G) is a 1-morphism from $*$ to (BG, c) . By [Proposition 2.1](#) it induces a boundary theory $1 \rightarrow \sigma_{BG,c}^{d+1}$. Composing the correspondence with the pointing $(BG, c) \leftarrow * \rightarrow *$ gives (X, τ) , so pairing this boundary theory with $\rho_{BG,c}$ gives $\sigma_{X,\tau}^d$. This is a $(\sigma_{BG,c}^{d+1}, \rho_{BG,c})$ -module structure, and by [Proposition 2.2](#) and [VD23] it is equivalently an anomalous theory defined on $\mathbf{Bord}_d^{BG}$, with anomaly theory classified by c . $\square$

Recall our d -dimensional theories $\sigma_{X,\tau}^d$ are valued in $\Omega\mathcal{T}$ (where $\mathcal{T}$ is fixed at the beginning of [Section 2](#)). Consider the d -type

$$B \text{Aut}_{\mathcal{T}}(\sigma_{X,\tau}^d(*))$$

with top homotopy group π_d . Because we assumed $\Omega^{d+1}\mathcal{T} \simeq \mathbf{k}$, scalars give a map:

$$(3.3) \quad \iota: B^d \mathbf{k}^\times \rightarrow B^d \pi_d.$$

Note that $B^d \pi_d$ is the fiber of the top stage $B \text{Aut}_{\mathcal{T}}(\sigma_{X,\tau}^d(*)) \rightarrow \pi_{\leq d-1} B \text{Aut}_{\mathcal{T}}(\sigma_{X,\tau}^d(*))$ of the Postnikov tower. The map ι is an equivalence when the invertible d -fold endomorphisms of the identity of $\sigma_{X,\tau}^d(*)$ are exactly the scalars, e.g. for a fusion category when $d = 3$. Write the k -invariant of $B \text{Aut}_{\mathcal{T}}(\sigma_{X,\tau}^d(*))$ over the truncation $\pi_{\leq d-1}$ as

$$(3.4) \quad k: \pi_{\leq d-1} B \text{Aut}_{\mathcal{T}}(\sigma_{X,\tau}^d(*)) \rightarrow B^{d+1} \pi_d.$$

Unpacking [Theorem 3.1](#) on the point gives the following.

Corollary 3.1. Assume [Hypothesis Q](#) and the hypotheses of [Theorem 3.1](#), and let c be as in [Theorem 3.1](#). Then there is a map

$$\mathbf{f}: BG \rightarrow \pi_{\leq d-1} B \operatorname{Aut}_{\mathcal{T}} (\sigma_{X,\tau}^d (*))$$

such that the projectivity c agrees with the pullback of the k -invariant:

$$\begin{array}{ccc} BG & \xrightarrow{\mathbf{f}} & \pi_{\leq d-1} B \operatorname{Aut}_{\mathcal{T}} (\sigma_{X,\tau}^d (*)) \\ \downarrow c & & \downarrow k \\ B^{d+1} \mathbf{k}^{\times} & \xrightarrow{B\iota} & B^{d+1} \pi_d \end{array}$$

where ι is from [\(3.3\)](#) and k is from [\(3.4\)](#).

Proof. By [Theorem 3.1](#), $\sigma_{X,\tau}^d$ is a projective BG -theory with projectivity c . By the cobordism hypothesis, a projective action of G with projectivity c on the object $x = \sigma_{X,\tau}^d (*)$ is the same as a map $\mathbf{f}: BG \rightarrow \pi_{\leq d-1} B \operatorname{Aut}_{\mathcal{T}} (x)$ together with a homotopy between $k \circ \mathbf{f}$ and $B\iota \circ c$. This is the usual obstruction theory for lifting through the top stage of the Postnikov tower (compare [\[VD23\]](#)). $\square$

4. SYMMETRIES OF THE BASE

Let L be a finite group, and τ a group cocycle representing a class in $H^3(L, \mathbf{k}^{\times})$. Let $\mathcal{C} = \mathbf{Vect}[L]^{\tau}$ be the fusion category of vector spaces graded by L , twisted by τ .

In [Section 4.1](#), we will construct an action by bimodules up to isomorphism:

$$f: G \rightarrow \pi_0 \operatorname{Aut}_{\mathbf{Fus}}(\mathcal{C})$$

which is induced by an action of G on the base:

$$\varphi: G \rightarrow \operatorname{Aut}(L, \tau)$$

where $\operatorname{Aut}(L, \tau)$ consists of group automorphisms of L which preserve the cohomology class of τ . This action can be twisted by a class γ satisfying a certain equation.

The map f lifts to a map to the full 3-group

$$\mathbf{f}: G \rightarrow \operatorname{Aut}_{\mathbf{Fus}}(\mathcal{C})$$

once we choose the further data μ and α of [Section 4.2](#) and [Section 4.3](#). All together, given L and τ , the map $\mathbf{f}$ is determined by φ and the triple (γ, μ, α) . Such a triple is the same thing as an extension of τ to a 3-cocycle on $L \rtimes_{\varphi} G$, and it need not exist: the obstructions are the differentials $d_2[\tau]$ and $d_3[\tau]$ in the Lyndon-Hochschild-Serre spectral sequence ([Proposition 4.1](#)). When φ preserves τ on the nose (e.g. when τ is trivial), the trivial triple works, so φ canonically defines such a map $\mathbf{f}$. This is the content of [Theorem 4.1](#).

In [Section 4.4](#), we describe a semiclassical action ([Definition 3.1](#)) associated to the same data $(\varphi, \gamma, \mu, \alpha)$. We expect that the quantization of this semiclassical action (i.e. its image under the quantization functor [\(2.5\)](#)) agrees with the map $\mathbf{f}$, but we do not prove this here.

4.1. Action via bimodules up to isomorphism. Consider an arbitrary group homomorphism

$$\varphi: G \rightarrow \operatorname{Aut}(L, \tau)$$

and a group 1-cochain on G valued in group 2-cochains on L :

$$(4.1) \quad \gamma: G \rightarrow C_{\text{gp}}^2(L, \mathbf{k}^{\times}) .$$

Here and below we write $\varphi(g)^* c$ for the pullback of a cochain c on L along the automorphism $\varphi(g)$, i.e. $(\varphi(g)^* c)(\ell_1, \dots, \ell_q) = c(\varphi(g)\ell_1, \dots, \varphi(g)\ell_q)$, and $\varphi(g)(\tau) := \varphi(g)^* \tau$. Suppose γ satisfies

$$(4.2) \quad d_L(\gamma(g)) = \varphi(g)^* \tau \cdot \tau^{-1}$$

for all $g \in G$, where d_L is the group differential for L . Then the graph $\Gamma(\varphi(g)) = \{(\ell, \varphi(g)\ell)\} \simeq L$, decorated by $\gamma(g)$, defines a 1-morphism in $\mathbf{Fam}_3(B^3\mathbf{k}^\times)$ from (BL, τ) to itself:

$$(4.3) \quad \begin{array}{ccc} & (B\Gamma(\varphi(g)), \gamma(g)) & \\ \swarrow & & \searrow \\ (BL, \tau) & & (BL, \tau) \end{array}$$

Indeed, (4.2) is the condition (2.3) for the legs $\ell \mapsto \ell$ and $\ell \mapsto \varphi(g)\ell$. Such a $\gamma(g)$ exists because $\varphi(g)$ preserves the cohomology class of τ , and the set of such $\gamma(g)$ is a torsor over $Z_{\text{grp}}^2(L, \mathbf{k}^\times)$. Write $C_\gamma(g)$ for this 1-morphism.

Equivalently, $\gamma(g)$ is a tensor structure on the functor $F_g: \mathcal{C} \rightarrow \mathcal{C}$, $\mathbf{k}_\ell \mapsto \mathbf{k}_{\varphi(g)\ell}$ (with the conventions matching (2.3)), so F_g is a tensor autoequivalence of $\mathcal{C}$. We define

$$(4.4) \quad f(g) := \mathcal{C}_{F_g}$$

to be the invertible $\mathcal{C}$ -bimodule category $\mathcal{C}$ with right action twisted by F_g . We expect that $f(g) \simeq \text{Sum}_3(C_\gamma(g))$, which would follow from the natural extension of Hypothesis Q' to twisted groupoids, and this is the motivation for the constructions below.

The $\mathcal{C}$ -bimodule $f(g)$ has the following description. The underlying category is the same as $\mathcal{C}$, and the left $\mathcal{C}$ -module structure is simply the tensor product in $\mathcal{C}$. The right action, say by some simple $\mathbf{k}_\ell \in \mathcal{C}$, is given by tensor product with $\mathbf{k}_{\varphi(\ell)}$.

The twist of this construction by the 2-cocycle $\gamma(g)$ can be thought of as follows. Rather than defining $f(g)$ to be the trivial $\mathcal{C}$ -bimodule twisted by a group automorphism of L , as above, we might consider twisting by a tensor autoequivalence Φ of $\mathcal{C}$. I.e. now an object $X \in \mathcal{C}$ acts on the right by tensoring with $\Phi(X)$. The tensor autoequivalences form an extension:

$$1 \rightarrow H^2(L, \mathbf{k}^\times) \rightarrow \pi_0 \text{Aut}_\otimes(\mathcal{C}) \rightarrow \text{Aut}(L, \tau) \rightarrow 1,$$

where $\text{Aut}(L, \tau)$ denotes the group automorphisms of L preserving the cohomology class defined by the cocycle τ . See [ENO10, §11.2] for more. The assignment $F \mapsto \mathcal{C}_F$ defines a homomorphism $\pi_0 \text{Aut}_\otimes(\mathcal{C}) \rightarrow \pi_0 \text{Aut}_{\mathbf{Fus}}(\mathcal{C})$. For nonabelian L it is not injective: conjugation by an invertible object is a tensor autoequivalence (lifting an inner automorphism of L) whose bimodule category is equivalent to $\mathcal{C}$.

4.2. System of products. Now we construct a system of products, i.e. equivalences

$$m(g, h) : f(g) \circ f(h) \xrightarrow{\sim} f(gh)$$

for all $g, h \in G$, where composites are written in the order of composition of functions, as in Section 5.6.

The composite $F_g \circ F_h$ is given by $\varphi(gh)$ on simple objects, with tensor structure $\gamma(h) \cdot \varphi(h)^* \gamma(g)$. Correspondingly, the composite of the correspondences $C_\gamma(h)$ and $C_\gamma(g)$ (in this order) is the graph of $\varphi(gh)$ decorated by $\gamma(h) \cdot \varphi(h)^* \gamma(g)$, since the decoration of the second correspondence is pulled back along $\varphi(h)$. Write

$$(d_G \gamma)(g, h) := \gamma(h) \cdot \varphi(h)^* \gamma(g) \cdot \gamma(gh)^{-1},$$

which is the group differential for G acting on $C_{\text{grp}}^2(L, \mathbf{k}^\times)$ by pullback along φ . Consider a group 2-cochain on G valued in 1-cochains on L

$$(4.5) \quad \mu : G^2 \rightarrow C_{\text{grp}}^1(L, \mathbf{k}^\times)$$

satisfying

$$(4.6) \quad d_L \mu = (d_G \gamma)^{-1} .$$

This is exactly the condition that $\mu(g, h)$, with component $\mu(g, h)(\ell)$ at $\mathbf{k}_\ell$, is a monoidal natural isomorphism $F_g \circ F_h \Rightarrow F_{gh}$ (again with the conventions matching (2.3)). We let $m(g, h)$ be the induced equivalence of bimodule categories:

$$(4.7) \quad m(g, h) : f(g) \circ f(h) \simeq \mathcal{C}_{F_g \circ F_h} \xrightarrow{\mu(g, h)} \mathcal{C}_{F_{gh}} = f(gh) .$$

In terms of correspondences, $\mu(g, h)$ defines the 2-morphism

$$(4.8) \quad \begin{array}{ccc} & (B\Gamma(\varphi(gh)), \mu(g, h)) & \\ \swarrow & & \searrow \\ (B\Gamma(\varphi(gh)), \gamma(h) \cdot \varphi(h)^* \gamma(g)) & & (B\Gamma(\varphi(gh)), \gamma(gh)) \end{array}$$

in $\mathbf{Fam}_3(B^3 \mathbf{k}^\times)$, which we write as $C_\mu(g, h)$, and we expect that $m(g, h) \simeq \text{Sum}_3(C_\mu(g, h))$. A μ satisfying (4.6) exists if and only if the class of $(d_G \gamma)(g, h)$ in $H_{\text{grp}}^2(L, \mathbf{k}^\times)$ vanishes for all g, h .

4.2.1. The degree three obstruction. Recall from [ENO10] the obstruction $O_3(f) \in H_{\text{grp}}^3(G, \pi_1)$ to the existence of a coherent system of products, where $\pi_1 = \pi_1 \text{Aut}_{\mathbf{Fus}}(\mathcal{C})$ is the group of isomorphism classes of invertible objects of $\mathcal{Z}(\mathcal{C})$. For τ trivial, $\pi_1 \simeq Z(L) \times L^\vee$, where $L^\vee = \text{Hom}(L, \mathbf{k}^\times) = Z_{\text{grp}}^1(L, \mathbf{k}^\times)$. In general π_1 contains $L^\vee$, as the invertible objects supported at the unit.

For the data (γ, μ) considered here, note that $d_L(d_G \mu) = d_G(d_L \mu) = d_G(d_G \gamma)^{-1} = 1$ by (4.6), so

$$d_G \mu \in Z_{\text{grp}}^3(G, L^\vee) ,$$

where $d_G \mu$ is the differential defined in Section 4.3.

Remark 10. This section is similar to the discussion of the same obstruction in [CGPW16, §5.1].

Proposition 4.1. *Fix φ and τ , and consider the Lyndon-Hochschild-Serre spectral sequence of $L \hookrightarrow L \rtimes_\varphi G \twoheadrightarrow G$ (see Section 4.4).*

- (i) *There exist γ satisfying (4.2) and μ satisfying (4.6) if and only if $d_2[\tau] = 0 \in E_2^{2,2} = H_{\text{grp}}^2(G, H_{\text{grp}}^2(L, \mathbf{k}^\times))$; for a given γ , $d_2[\tau]$ is represented by $(g, h) \mapsto [(d_G \gamma)(g, h)]$.*
- (ii) *Given these, γ and μ can be chosen so that moreover $d_G \mu = 1$ if and only if $d_3[\tau] = 0$ in $E_3^{3,1}$, a subquotient of $H_{\text{grp}}^3(G, L^\vee)$, where $d_3[\tau]$ is represented by $[d_G \mu]$.*

These obstructions need not vanish in general. They vanish, for instance, if $\varphi(g)^ \tau = \tau$ for all g (take $\gamma = 1$ and $\mu = 1$), or if $|G|$ and $|L|$ are coprime, since then $E_2^{p,q} = 0$ for $p, q > 0$.*

Proof. The equations (4.2), (4.6) and $d_G \mu = 1$ are the cocycle conditions in total degree 3 of the double complex $C_{\text{grp}}^p(G, C_{\text{grp}}^q(L, \mathbf{k}^\times))$, with G acting through φ , for an element whose $(0, 3)$ -component is τ . Since the extension is split, G acts on the complex $C_{\text{grp}}^*(L, \mathbf{k}^\times)$, and this double complex computes $H_{\text{grp}}^*(L \rtimes_\varphi G, \mathbf{k}^\times)$ [HS53]. The spectral sequence of its filtration by p is the Lyndon-Hochschild-Serre spectral sequence, and the statements are the usual description of the differentials on $E_r^{0,3}$ by zig-zags in the double complex. $\square$

4.3. Associators. Now we construct associators, i.e. invertible modifications

$$a(g, h, k) : m(gh, k) \circ (m(g, h) \circ \text{id}_{f(k)}) \xrightarrow{\sim} m(g, hk) \circ (\text{id}_{f(g)} \circ m(h, k))$$

between the two equivalences $f(g) \circ f(h) \circ f(k) \simeq f(ghk)$. In terms of correspondences, the source and target are the identity correspondence on $B\Gamma(\varphi(ghk))$ decorated, respectively, by

$$\varphi(k)^* \mu(g, h) \cdot \mu(gh, k) \quad \text{and} \quad \mu(h, k) \cdot \mu(g, hk) ,$$

where $\mu(g, h)$ is pulled back along $\varphi(k)$ because it is whiskered with $f(k)$, which is applied first. Write

$$(d_G \mu)(g, h, k) := \varphi(k)^* \mu(g, h) \cdot \mu(gh, k) \cdot \mu(g, hk)^{-1} \cdot \mu(h, k)^{-1} ,$$

which is the group differential for G acting on $C_{\text{grp}}^1(L, \mathbf{k}^\times)$ by pullback along φ . (One checks directly that $d_G(d_G \gamma) = 1$, and d_G commutes with d_L .) A 3-morphism between these correspondences is given by a 0-cochain, i.e. a scalar

$$(4.9) \quad \alpha(g, h, k) \in C_{\text{grp}}^0(L, \mathbf{k}^\times) = \mathbf{k}^\times ,$$

satisfying

$$(4.10) \quad d_L \alpha(g, h, k) = (d_G \mu)(g, h, k)^{-1} .$$

Since L acts trivially on $\mathbf{k}^\times$, $d_L \alpha = 1$, so this reduces to

$$(4.11) \quad d_G \mu = 1 .$$

Equivalently, the two composite natural isomorphisms $F_g \circ F_h \circ F_k \Rightarrow F_{ghk}$ built from μ agree. Given this, the scalar $\alpha(g, h, k)$ defines the modification

$$(4.12) \quad a(g, h, k) := \alpha(g, h, k) \cdot \text{id} ,$$

which we expect to be the image under Sum_3 of the 3-morphism $C_\alpha(g, h, k)$ given by $B\Gamma(\varphi(ghk))$ decorated by $\alpha(g, h, k)$.

4.3.1. Degree four obstruction. Suppose γ and μ satisfy (4.2), (4.6) and (4.11). Then the pentagon condition on the associators (4.12) is $d_G \alpha = 1$, i.e. $\alpha \in Z_{\text{grp}}^3(G, \mathbf{k}^\times)$. In particular $\alpha = 1$ works, so the obstruction $O_4(f, m) \in H_{\text{grp}}^4(G, \mathbf{k}^\times)$ of [ENO10] vanishes, and different choices of α form the usual $H_{\text{grp}}^3(G, \mathbf{k}^\times)$ -torsor. In terms of Proposition 4.1, this reflects the vanishing of $d_4[\tau] \in E_4^{4,0}$, which holds because $L \rtimes_\varphi G \rightarrow G$ splits: $H_{\text{grp}}^*(G, \mathbf{k}^\times) \rightarrow H_{\text{grp}}^*(L \rtimes_\varphi G, \mathbf{k}^\times)$ is injective, so no differential hits the bottom row.

4.3.2. Fully coherent action induced by an automorphism of the base.

Theorem 4.1. *Let L and G be finite groups, let $\tau \in Z_{\text{grp}}^3(L, \mathbf{k}^\times)$, and let $\varphi : G \rightarrow \text{Aut}(L, \tau)$ be a homomorphism, where $\text{Aut}(L, \tau)$ denotes the group of automorphisms preserving the cohomology class of τ .*

- (i) *A triple (γ, μ, α) , with $\gamma \in C_{\text{grp}}^1(G, C_{\text{grp}}^2(L, \mathbf{k}^\times))$, $\mu \in C_{\text{grp}}^2(G, C_{\text{grp}}^1(L, \mathbf{k}^\times))$ and $\alpha \in Z_{\text{grp}}^3(G, \mathbf{k}^\times)$, satisfying (4.2), (4.6) and (4.11), defines a map of 3-groups*

$$(4.13) \quad \mathbf{f} = (f, m, a) : G \rightarrow \text{Aut}_{\mathbf{Fus}}(\mathbf{Vect}[L]^\tau)$$

with f, m, a as in (4.4), (4.7), (4.12). Assuming Hypothesis Q, this is equivalently a (σ_{BG}^4, ρ) -module structure on $\sigma_{BL, \tau}^3$.

- (ii) *Such a triple exists if and only if $[\tau]$ is in the image of the restriction $H_{\text{grp}}^3(L \rtimes_\varphi G, \mathbf{k}^\times) \rightarrow H_{\text{grp}}^3(L, \mathbf{k}^\times)$, if and only if $d_2[\tau] = 0$ and $d_3[\tau] = 0$ (Proposition 4.1). This holds, for example, if $|G|$ and $|L|$ are coprime.*
- (iii) *If $\varphi(g)^* \tau = \tau$ for all g (for example, if τ is trivial), then $(1, 1, 1)$ is such a triple, so φ canonically defines such a module structure.*

Proof. For (i): by (4.2), (4.6) and (4.11), the tensor autoequivalences F_g and the isomorphisms $\mu(g, h)$ define an action of G on $\mathcal{C} = \mathbf{Vect}[L]^\tau$ by tensor autoequivalences. The crossed product $\mathcal{C} \rtimes G$, with associativity constraint twisted by α , is a faithfully G -graded fusion category with trivial component $\mathcal{C}$ and g -component $\mathcal{C}_{F_g} = f(g)$ [Tam01, Gal11]. By [ENO10, Theorem 1.3], such a graded extension is the same as a map $BG \rightarrow B\mathrm{Aut}_{\mathbf{Fus}}(\mathcal{C})$, whose underlying data are (f, m, a) as in Sections 4.1 to 4.3. By the cobordism hypothesis this classifies a theory $\mathbf{Bord}_3^{BG} \rightarrow \mathbf{Fus}$, and by Proposition 2.2 this is equivalently a (σ_{BG}^4, ρ) -module structure on $\sigma_{BL, \tau}^3$.

For (ii): by the proof of Proposition 4.1, triples are the same as 3-cocycles in the total complex whose $(0, 3)$ -component is τ , and the double complex computes the cohomology of $L \rtimes_\varphi G$, with restriction to L given by projection onto the $(0, 3)$ -component. Since $L \rtimes_\varphi G \rightarrow G$ splits, $d_4[\tau] = 0$ automatically (Section 4.3.1). In the coprime case, $E_2^{p,q} = 0$ for $p, q > 0$.

For (iii): if $\varphi(g)^* \tau = \tau$ then $\gamma = 1$ satisfies (4.2), so $d_G \gamma = 1$, and $\mu = 1$ and $\alpha = 1$ satisfy the remaining conditions. $\square$

Remark 11. We consider semiclassical actions $BG \leftarrow C_G \rightarrow *$ of G on BL (Definition 3.1) of a very special form in this section:

$$C_G = B(L \rtimes_\varphi G) .$$

Arbitrary such semiclassical actions C_G are equivalent to arbitrary group extension of G by L , which are classified by degree 2 nonabelian group cohomology of G with coefficients in L .

Even if L is abelian, extensions given by semidirect products are still quite restrictive: For every φ , there is an extension associated to every class in $H_{\mathrm{Grp}, \varphi}^2(G, L)$, where this is group cohomology with the action φ of G on L .

Such, more general, considerations are made in [ENO10, §9.1, §11]. Namely, one can start from a homomorphism $\varphi: G \rightarrow \mathrm{Out}(L)$ and lifts $\tilde{\varphi}(g) \in \mathrm{Aut}(L)$. Then $\tilde{\varphi}(g)\tilde{\varphi}(h)\tilde{\varphi}(gh)^{-1}$ is conjugation by some $n(g, h) \in L$, and when the resulting obstruction in $H_{\mathrm{grp}}^3(G, Z(L))$ vanishes, suitable choices of n define a group extension $1 \rightarrow L \rightarrow E \rightarrow G \rightarrow 1$, which replaces $L \rtimes_\varphi G$. In this non-split case $d_4[\tau] \in E_4^{4,0}$ need not vanish, and gives a possibly nonzero obstruction O_4 .

4.4. Semiclassical action. In this section we will construct a semiclassical action of G on (BL, τ) , from the same data as above: an action φ and a triple (γ, μ, α) . We expect that the quantization of this semiclassical action (i.e. its image under the quantization functor Sum_4 in (2.5)) is equivalent to the map (f, m, a) in (4.13).

Note that the map φ canonically defines a semiclassical action of G on BL (Example 2), and recall the anomaly/obstruction to the existence of a semiclassical action of G on (BL, τ) studied in Section 3.2. In this case, the space X is BL , so the Serre spectral sequence considered in Section 3.2 reduces to the Lyndon-Hochschild-Serre (LHS) spectral sequence associated to the normal subgroup $L \hookrightarrow L \rtimes_\varphi G$:

$$E_2^{p,q} = H_{\mathrm{Grp}}^p(G, H^q(L, \mathbf{k}^\times)) \Rightarrow H_{\mathrm{Grp}}^{p+q}(L \rtimes_\varphi G, \mathbf{k}^\times) .$$

See [Mü20, MS20, KT14] and [ENO10, §11.8] for more on the LHS spectral sequence as a tool for encoding anomalies.

The following follows from the proof of Proposition 4.1 (see also [ENO10, §11.8]).

Proposition 4.2. *A triple (γ, μ, α) as in Theorem 4.1 is the same as a 3-cocycle in the total complex of $C_{\mathrm{grp}}^p(G, C_{\mathrm{grp}}^q(L, \mathbf{k}^\times))$ with $(0, 3)$ -component τ , and so determines a 3-cocycle τ_G on $L \rtimes_\varphi G$ restricting to τ on L , up to coboundaries of cochains that vanish on L . Every such τ_G arises in this way.*

Therefore, we have a 1-morphism in the category $\mathbf{Fam}_4(B^4\mathbf{k}^\times)$:

$$\begin{array}{ccc} & (B(L \rtimes_\varphi G), \tau_G) & \\ \swarrow & & \searrow \\ BG & & * \end{array}$$

By [Proposition 2.1](#), this induces a domain wall

$$F_G: 1 \rightarrow \sigma_{BG}^4,$$

such that pairing with the regular boundary condition recovers $\sigma_{BL,\tau}^3$. By [Proposition 2.2](#), F_G is equivalent to a representation of $\mathbf{Bord}_3^{BG}$, and by the cobordism hypothesis to a functor

$$F_G: BG \rightarrow B \operatorname{Aut}_{\mathbf{Fus}}(\mathcal{C}).$$

We expect that F_G is equivalent to the map (f, m, a) of [Theorem 4.1](#), but we do not prove this here.

5. ORTHOGONAL SYMMETRIES

5.1. Canonical kernels. Let (A, q) be a finite abelian metric group (i.e. a finite abelian group equipped with nondegenerate quadratic form $q: A \rightarrow \mathbf{k}^\times$) with two Lagrangian subgroups $L, K \subset A$. Then we have a correspondence of groups (here and below, in correspondences we write a group for its classifying space, e.g. L for BL):

$$\begin{array}{ccc} & A/(L \cap K) & \\ \swarrow & & \searrow / (L/L \cap K) \\ A/K & & A/L \end{array}$$

If $L \cap K = \{1\}$, i.e. they are transverse, then this takes the simpler form:

$$(5.1) \quad \begin{array}{ccc} & A & \\ \swarrow & & \searrow \\ L \simeq A/K & & A/L \simeq K \end{array}$$

Decorating the correspondence (5.1) with any 2-cocycle κ on A defines a 1-morphism in $\mathbf{Fam}_3(B^3\mathbf{k}^\times)$ (recall [Section 2.3](#)). We will primarily make use of the correspondence (5.1) when K is the image of L under some $g \in \operatorname{O}(A, q)$. Continue to assume that $gL \cap L$ is trivial, so that (5.1) decorated by κ is the 1-morphism

$$\begin{array}{ccc} & (A, \kappa) & \\ \swarrow & & \searrow \\ L & & gL \end{array}$$

in $\mathbf{Fam}_3(B^3\mathbf{k}^\times)$.

Assume there exists a polarization $A \simeq L \oplus L^\vee$, and that q is given by evaluation:

$$\operatorname{ev}: A \simeq L \oplus L^\vee \rightarrow \mathbf{k}^\times.$$

There is a canonical class in $H^2(L \oplus L^\vee)$ corresponding to the Heisenberg extension. Let $\kappa_{\operatorname{ev}}$ be any 2-cocycle representing the canonical class. For instance, one particular 2-cocycle representing this class sends a pair of elements (ℓ, φ) and (k, ψ) of $L \oplus L^\vee$ to $\psi(\ell)$. This defines a 1-morphism in $\mathbf{Fam}_3(B^3\mathbf{k}^\times)$:

$$(5.2) \quad \begin{array}{ccc} & (A, \kappa_{\operatorname{ev}}) & \\ \swarrow p_1 & & \searrow p_2 \\ L & & gL \end{array}$$

for any $g \in O(L \oplus L^\vee)$ such that $gL \cap L$ is trivial.

Remark 12. This cocycle κ_{ev} should be thought of as an analogue of the BKS kernel in ordinary geometric quantization [LV80].

5.2. Higher intertwining operators. The correspondences built in [Section 5.1](#) induce domain walls between the associated TQFTs. Assume [Hypothesis Q](#) so that there are well-defined TQFTs

$$\sigma_{BL}^3: \mathbf{Bord}_3^{\text{fr}} \rightarrow \mathbf{Fus}$$

for any finite abelian group L defined using the functor Sum_3 .

Let $g \in O(L \oplus L^\vee)$ such that $gL \cap L$ is trivial. Applying [Proposition 2.1](#) to (5.2), we obtain a domain wall:

$$\sigma_{BL}^3 \rightarrow \sigma_{B(gL)}^3 .$$

Let

$$\iota: gL \xrightarrow{\sim} A/L = L^\vee$$

be the canonical isomorphism, given by the inclusion $gL \hookrightarrow A$ followed by the projection $A \rightarrow A/L$, where we identify $A/L = L^\vee$ using the polarization. Composing (5.2) with ι gives the following correspondence:

$$(5.3) \quad \begin{array}{ccc} & (A, \kappa_{\text{ev}}) & \\ \swarrow p_1 & & \searrow \iota \circ p_2 \\ L & & L^\vee \end{array}$$

Definition 5.1. The twice-categorified Fourier transform associated to g is the domain wall obtained by applying [Proposition 2.1](#) to the correspondence (5.3):

$$\text{FT}_g: \sigma_{BL}^3 \rightarrow \sigma_{BL^\vee}^3 .$$

If $gL = L^\vee$ (e.g. $g = T_\sigma$ as in [Section 5.3](#)), then ι is the identity, the two legs of (5.3) are the projections of $A = L \oplus L^\vee$, and FT_g does not depend on g . In this case we write $\text{FT} := \text{FT}_g$.

Remark 13. Like the classical Fourier transform $L^2(\ell) \rightarrow L^2(\ell^*)$, the wall FT depends only on the polarization $A = L \oplus L^\vee$. The choice of a symmetric isomorphism $\sigma: L \simeq L^\vee$ only enters when FT is regarded as an autoequivalence of σ_{BL}^3 , as in (5.7).

For general g with $gL \cap L$ trivial, the Lagrangian gL is transverse to L , so it is the graph $\{(\beta\varphi, \varphi)\}$ of a homomorphism $\beta: L^\vee \rightarrow L$ with $\varphi(\beta\varphi) = 1$ for all φ . Thus $gL = U_\beta L^\vee$, where $U_\beta(\ell, \varphi) = (\ell + \beta\varphi, \varphi)$ is orthogonal and preserves L . Then $T_\sigma U_\beta^{-1}g$ preserves L , so $g = U_\beta T_\sigma p$ with U_β and p in the stabilizer of L . So, up to composing with elements of this stabilizer, it suffices to consider $g = T_\sigma$, for which κ_{ev} is the kernel attached to the pair $(L, gL) = (L, L^\vee)$. Compare [Lemma 5.1](#), where the roles of L and $L^\vee$ are exchanged.

5.3. Evaluation on the point.

5.3.1. As a $\mathbf{Vect}[L]$ - $\mathbf{Rep}(L)$ -bimodule. Recall from [Example 1 \(Item \(iii\)\)](#) the TQFT σ_{BA}^3 for any abelian group sends the point to the fusion category $\mathbf{Vect}[A]$. The natural transformation FT, evaluated on the point, is therefore a $\mathbf{Vect}[L]$ - $\mathbf{Vect}[L^\vee]$ -bimodule category. As it turns out (shown in [Proposition 5.1](#)) this is an invertible bimodule category.

Recall that there is an equivalence of monoidal categories:

$$(5.4) \quad \mathbf{Rep}(L) \cong \mathbf{Vect}[L^\vee]$$

given by spectrally decomposing representations of L . Therefore $\text{FT}(\ast)$ is equivalently an invertible $\mathbf{Vect}[L]$ - $\mathbf{Rep}(L)$ -bimodule category. Recall that here $\mathbf{Rep}(L)$ has the symmetric-monoidal tensor product, whereas the monoidal structure on $\mathbf{Vect}[L]$ is given by convolution. In other words:

Proposition 5.1. *The twice-categorified Fourier transform (evaluated on a point) is a Morita equivalence between $\mathbf{Vect}[L]$ and $\mathbf{Rep}(L)$, i.e it exchanges convolution with the symmetric-monoidal tensor product. It also has a single simple object, i.e. is $\mathbf{Vect}$.*

Proof. Recall the classification of $\mathbf{Vect}[L]\text{-}\mathbf{Vect}[L^\vee]$ -bimodule categories is equivalent to the classification of $\mathbf{Vect}[L \oplus L^\vee]$ -module categories. The latter are explicitly classified, e.g. in [ENO10] by subgroups of $L \oplus L^\vee$ equipped with alternating bicharacters. In particular, the module category (and therefore bimodule category) corresponding to a given subgroup H has $|L \oplus L^\vee|/|H|$ simple objects. In this case the subgroup H is the entire group, so there is a single simple object.

The fact that this is an equivalence follows from the invertibility of the bimodule category, which follows from Proposition 5.2. $\square$

Remark 14. The Morita equivalence in Proposition 5.1 already appears in [EGNO15, Example 7.12.19] and [FT22, Proposition 3.14].

5.3.2. *As a $\mathbf{Vect}[L]$ -bimodule.* Choose some group isomorphism⁸

$$(5.5) \quad \sigma: L \xrightarrow{\sim} L^\vee$$

such that

$$(5.6) \quad \sigma(\ell)(k) = \sigma(k)(\ell)$$

for all $k, \ell \in L$. Define the corresponding orthogonal transform $T_\sigma \in \mathrm{O}(L \oplus L^\vee)$ by

$$T_\sigma = \begin{pmatrix} 0 & \sigma^{-1} \\ \sigma & 0 \end{pmatrix}$$

Note that this preserves the quadratic form because (5.6) is satisfied:

$$\mathrm{ev}(T_\sigma(\ell, \varphi)) = \sigma(\ell)[\sigma^{-1}(\varphi)] = \sigma(\sigma^{-1}\varphi)(\ell) = \varphi(\ell) = \mathrm{ev}(\ell, \varphi)$$

Remark 15. This is a noncanonical choice! For any abelian group L , such a symmetric equivalence σ always exists, which can be seen e.g. componentwise in the primary decomposition of L . If σ' is isometric to σ , i.e. $\sigma'(u\ell)(uk) = \sigma(\ell)(k)$ for some $u \in \mathrm{Aut}(L)$, then a direct computation gives $T_{\sigma'} = D_u T_\sigma D_u^{-1}$, where $D_u(\ell, \varphi) = (u\ell, \varphi \circ u^{-1})$. In general, however, T_σ and $T_{\sigma'}$ need not be conjugate in $\mathrm{O}(L \oplus L^\vee)$: for $L = \mathbb{Z}/p$ with p odd, they are conjugate if and only if σ'/σ is a square in $\mathbb{F}_p^\times$ (Example 4).

Composing the correspondence (5.3) with the equivalence (5.5), we obtain a correspondence:

$$(5.7) \quad \begin{array}{ccc} & (L \oplus L^\vee, \kappa_{\mathrm{ev}}) & \\ \swarrow p_1 & & \searrow \sigma^{-1} \circ \iota \circ p_2 \\ L & & L \end{array}$$

Equivalently, we can consider the correspondence:

$$(5.8) \quad \begin{array}{ccc} & (L \oplus L, \kappa_L) & \\ \swarrow p_1 & & \searrow p_2 \\ L & & L \end{array}$$

where κ_L is a 2-cocycle on L^2 representing the class in $H_{\mathrm{Grp}}^2(L^2, \mathbf{k}^\times)$ corresponding to the skew-symmetric bicharacter $\mathrm{alt} \kappa_L$ (see Section A) sending

$$(5.9) \quad (\ell_1, k_1), (\ell_2, k_2) \mapsto \sigma(k_2)(\ell_1)^{-1} \sigma(k_1)(\ell_2) .$$

⁸We use σ both for this isomorphism and, following [FMT22], for the theories σ_X ; the latter always carry a subscript.

We obtain this formula by post-composing the 2-cocycle κ_{ev} with σ and the inversion map, followed by the alternating map from (A.1) to obtain the alternating bicharacter. Write $C_{L,L}$ for the correspondence in (5.7) (or equivalently (5.8)).

Recall the quantization map Sum_3 from Hypothesis Q, in particular the explicit description of Sum_3 in Example 1 (Items (iii) and (iv)).

Proposition 5.2. *Assume Hypothesis Q'. The bimodule category $\text{Sum}_3(C_{L,L})$ has a single simple object, and is of order 2.*

Proof. The bimodule category $\text{Sum}_3(C_{L,L})$ corresponds to a $\mathbf{Vect}[L^2]$ -module category under the usual correspondence described in Section A.

Recall that module categories over $\mathbf{Vect}[L^2]$ are classified by subgroups of L^2 equipped with alternating bicharacters. The subgroup corresponding to $\text{Sum}_3(C_{L,L})$ is the whole group L^2 , which is enough to see that the (bi)module category has a single simple object, since simple objects are in correspondence with cosets of the subgroup in L^2 [ENO10, §2.7].

The alternating bicharacter is $\text{alt } \kappa_L$ defined in (5.9). Alternating bicharacters on an abelian group are equivalent to degree 2 cohomology classes on the group (see Section A). Recall the Schur isomorphism ((A.2)) in this case:

$$(5.10) \quad H^2(L^2) \simeq H^2(L) \oplus H^2(L) \oplus \text{Hom}(L \otimes_{\mathbb{Z}} L, \mathbf{k}^\times)$$

where the third factor consists of bicharacters on L . The cohomology class of κ_L is trivial in the first two factors, and in the third factor it is the bicharacter $(\ell, k) \mapsto \text{alt } \kappa_L((\ell, 0), (0, k)) = \sigma(k)(\ell)^{-1}$, by (5.9).

By Lemma A.1, the bimodule therefore has the following explicit description. The underlying category is $\mathbf{Vect}$ and the action of $\mathbf{Vect}[L]$ on either side is trivial. The only nontrivial data is the middle associator, which by Lemma A.1 is the inverse of the third component above, i.e. the bicharacter on L sending:

$$(\ell, k) \mapsto \sigma(k)(\ell) .$$

As is explained in [ENO10, Prop. 9.3], this bimodule category is invertible on account of the fact that σ is an isomorphism, and it is order 2 on account of the fact that σ satisfies (5.6). $\square$

5.4. Full orthogonal group. It is shown in [ENO10] that the group of invertible $\mathbf{Vect}[L]$ -bimodules is equivalent to the group $\text{O}(L \oplus L^\vee)$. This equivalence is given a description in [ENO10, §10] via the hyperbolic part of the category of Lagrangian correspondences.

Namely, they assign the following $\mathbf{Vect}[L^2]$ -module category to $g \in \text{O}(L \oplus L^\vee, \text{ev})$. Recall $\mathbf{Vect}[L^2]$ -module categories are classified by a subgroup $H \subset L^2$ and a class in $H^2(L^2, \mathbf{k}^\times)$, or equivalently an alternating bicharacter on L^2 (see Section A). The subgroup H is the projection of the graph of g to $L^2 \subset (L \oplus L^\vee)^2$. The alternating bicharacter χ_g is defined by:

$$\chi_g((\ell_1, k_1), (\ell_2, k_2)) := \left\langle \widetilde{(\ell_1, k_1)}, (\ell_2, 1, k_2, 1) \right\rangle_q$$

where $\widetilde{(\ell_1, k_1)}$ denotes an arbitrary lift of (ℓ_1, k_1) to the graph of g in $(L \oplus L^\vee)^2$ and $q = \text{ev}_A \oplus \text{ev}_A^{-1}$.

In particular, this bicharacter works out to be:

$$\begin{aligned} \chi_g((\ell_1, k_1), (\ell_2, k_2)) &= \left\langle \widetilde{(\ell_1, k_1)}, (\ell_2, 1, k_2, 1) \right\rangle_q \\ &= \frac{q(\ell_1 + \ell_2, \varphi, k_1 + k_2, \psi)}{q(\ell_1, \varphi, k_1, \psi)} \\ &= \varphi(\ell_2) \psi(k_2)^{-1} , \end{aligned}$$

where $\varphi \in L^\vee$ is such that

$$k_1 = p_3 g(\ell_1, \varphi) ,$$

where p_3 denotes the projection from the graph of g to the third factor L , and

$$\psi := p_4 g(\ell_1, \varphi) .$$

Recasting this in $\mathbf{Fam}_3(B^3\mathbf{k}^\times)$, there is a correspondence C_g defined for every element of $g \in \mathcal{O}(L \oplus L^\vee)$:

$$(5.11) \quad C_g = \begin{array}{ccc} & (p_{1,3}\Gamma(g), \kappa_g) & \\ \swarrow & & \searrow \\ L & & L \end{array}$$

where κ_g is a 2-cocycle representing a class in $H^2(p_{1,3}\Gamma(g))$ which corresponds to the alternating bicharacter χ_g on $p_{1,3}\Gamma(g)$ defined above.

Define a functor

$$(5.12) \quad f: \mathcal{O}(L \oplus L^\vee) \rightarrow \pi_0 \text{Aut}_{\mathbf{Fus}}(\mathbf{Vect}[L])$$

by sending g to $\text{Sum}_3(C_g)$. Recall (from [Example 1 Items \(iii\) and \(iv\)](#)) that the value of Sum_3 is given precisely by the equivalence given in [ENO10] between $\mathcal{O}(L \oplus L^\vee)$ and the group of isomorphism classes of invertible $\mathbf{Vect}[L]$ -bimodule categories.

Theorem 5.1. *Assume [Hypothesis Q](#) and [Hypothesis Q'](#). The map f (5.12) sends the order 2 element T_σ to $\text{Sum}_3(C_{L,L})$, i.e. to the wall FT of [Definition 5.1](#) followed by the identification $\sigma_{BL^\vee}^3 \simeq \sigma_{BL}^3$ induced by σ , as in (5.7).*

Proof. It suffices to show that the correspondence C_g in (5.11) and $C_{L,L}$ in (5.8) are equivalent 1-morphisms in $\mathbf{Fam}_3(B^3\mathbf{k}^\times)$, i.e. we just have to show they are decorated by the same 2-cocycle, or equivalently the same bicharacter.

As above, the bicharacter χ_g corresponding to C_g (the image under (A.1) of κ_g) is:

$$\chi_g((\ell_1, k_1), (\ell_2, k_2)) = \varphi(\ell_2) \psi(k_2)^{-1} .$$

When $g = T_\sigma$, a particular choice of lift is given by

$$\varphi = \sigma(k_1) \quad \psi = \sigma(\ell_1)$$

making the formula:

$$\chi_{T_\sigma}((\ell_1, k_1), (\ell_2, k_2)) = \sigma(k_1) \ell_2 \sigma(\ell_1) (k_2)^{-1} .$$

On the other hand, recall that the bicharacter defined in (5.9), corresponding to the cocycle decorating $C_{L,L}$ sends:

$$(\ell_1, k_1), (\ell_2, k_2) \mapsto \sigma(k_2) (\ell_1)^{-1} \sigma(k_1) (\ell_2) .$$

On account of (5.6) we have

$$\sigma(k_2) (\ell_1) = \sigma(\ell_1) (k_2) ,$$

and therefore the bicharacters agree. $\square$

5.5. Field-theoretic interpretation. As is explained in [VD23], the results of [ENO10] and [Proposition 2.2](#) together imply that the group of orthogonal automorphisms of $A \simeq L \oplus L^\vee$ canonically induce projective automorphisms of the theory σ_{BL}^3 . In other words: Twice-categorified geometric quantization of odd-order metric groups is projectively equivariant with respect to the orthogonal group.

Theorem 5.2. *Assume [Hypothesis Q](#) and [Hypothesis Q'](#) and that L has odd order. The canonical projective action of $T_\sigma \in \mathrm{O}(L \oplus L^\vee)$ on σ_{BL}^3 is via the domain wall FT ([Definition 5.1](#)) followed by the identification induced by σ , i.e. via $\mathrm{Sum}_3(C_{L,L})$.*

Proof. Recall that the canonical projective action of $\mathrm{O}(L \oplus L^\vee)$ on σ_{BL}^3 is defined as follows. In [ENO10, §5.2] an equivalence

$$\Psi: \mathrm{Aut}_{\mathbf{EqBr}}(\mathcal{Z}(\mathcal{C})) \xrightarrow{\sim} \pi_{\leq 1} \mathrm{Aut}_{\mathbf{Fus}}(\mathcal{C})$$

is provided for any fusion category $\mathcal{C}$. When $\mathcal{C} = \mathbf{Vect}[L]$ and $|L|$ is odd there is a canonical splitting ([EG18, §6] and [CGPW16, §5]):

$$s: \mathrm{O}(L \oplus L^\vee, \mathrm{ev}) = \pi_0 \mathrm{Aut}_{\mathbf{EqBr}}(\mathcal{Z}(\mathcal{C})) \rightarrow \pi_{\leq 1} \mathrm{Aut}_{\mathbf{EqBr}}(\mathcal{Z}(\mathcal{C})) .$$

Composing these maps yields a canonical action of $\mathrm{O}(L \oplus L^\vee)$ on $\mathbf{Vect}[L]$ regarded as an object of $\mathbb{P}\mathbf{Fus}$. By the cobordism hypothesis this classifies a projective action of $\mathrm{O}(L \oplus L^\vee)$ on σ_{BL}^3 . On isomorphism classes, $\Psi \circ s$ is the isomorphism f of (5.12) [ENO10, §10], so the theorem follows from the cobordism hypothesis and [Theorem 5.1](#). $\square$

Remark 16. The odd-order assumption is only needed for the canonical projective action to be defined. On isomorphism classes, [Theorem 5.1](#) identifies $f(T_\sigma)$ with $\mathrm{Sum}_3(C_{L,L})$ for any finite abelian group L .

5.6. Composition of kernels. In geometric quantization, the failure of the integral transforms T_g to compose is measured by the scalars $c(g, h)$ from the introduction. In this section we describe the analogous structure for the walls constructed above. Throughout, $A = L \oplus L^\vee$ with L a finite abelian group, we assume [Hypothesis Q](#) and [Hypothesis Q'](#), and f denotes the map (5.12), which is an isomorphism of groups by [ENO10]. We write $C_{L,L}^\sigma$ for the correspondence (5.8) defined using a symmetric isomorphism σ as in (5.5) and (5.6).

Conventions. By definition, $f(g) = \mathrm{Sum}_3(C_g)$, where C_g is the subgroup $p_{1,3}\Gamma(g) \subset L \times L$ (first factor the source, second the target) decorated by a 2-cocycle κ_g with $\mathrm{alt} \kappa_g = \chi_g$, and

$$\chi_g((\ell_1, k_1), (\ell_2, k_2)) = \varphi(\ell_2) \psi(k_2)^{-1}$$

for any φ with $g(\ell_1, \varphi) = (k_1, \psi)$ ([Section 5.4](#)). We write composites of correspondences, and of the walls they induce, in the order of composition of functions. Since $\Gamma(gh)$ is the composite of the relations $\Gamma(h)$ and $\Gamma(g)$, this is the order for which $f(gh) \simeq f(g) \circ f(h)$; this is also confirmed by the direct computations of composites in [Remark 17](#) and [Example 4](#). With these conventions, $C_{L,L}^\sigma$ and C_{T_σ} are decorated by the same alternating bicharacter (proof of [Theorem 5.1](#)).

5.6.1. The two extreme cells. Let $P \subset \mathrm{O}(A)$ denote the stabilizer of the Lagrangian $L^\vee$. For $u \in \mathrm{Aut}(L)$, and for an alternating bicharacter b on L , regarded as a homomorphism $b: L \rightarrow L^\vee$ with $b(\ell)(\ell) = 1$ for all ℓ , write

$$D_u(\ell, \varphi) = (u\ell, \varphi \circ u^{-1}) , \quad S_b(\ell, \varphi) = (\ell, \varphi \cdot b(\ell)) .$$

These preserve ev and $L^\vee$. Conversely, an element $g \in P$ has the form $g(\ell, \varphi) = (u\ell, c(\ell) \cdot (\varphi \circ u^{-1}))$, and preserving ev forces $c(\ell)(u\ell) = 1$, so $g = S_b D_u$ with $b = c \circ u^{-1}$. This decomposition is unique, so

$$P \simeq \wedge^2(L) \rtimes \mathrm{Aut}(L) \simeq H^2(L, \mathbf{k}^\times) \rtimes \mathrm{Aut}(L)$$

by (A.1).

By the description of f in Section 5.4, the kernel of the projection $p_{1,3}: \Gamma(g) \rightarrow L^2$ is $\{(a, ga) : a, ga \in L^\vee\} \simeq L^\vee \cap g^{-1}L^\vee$, so $f(g)$ has $|gL^\vee \cap L^\vee|$ simple objects. The two extreme cases are:

- $g \in P$, in which case $f(g)$ has $|L|$ simple objects. For $g = D_u$, taking $\varphi = 1$ gives $k_1 = u\ell_1$ and $\psi = 1$, so $p_{1,3}\Gamma(g) = \Gamma(u)$ and $\chi_g = 1$: $f(D_u)$ is the image of the (undecorated) graph of u . For $g = S_b$, taking $\varphi = 1$ gives $k_1 = \ell_1$ and $\psi = b(\ell_1)$, so the subgroup is the diagonal, decorated by a 2-cocycle γ with $\text{alt } \gamma(\ell_1, \ell_2) = b(\ell_1)(\ell_2)^{-1} = b(\ell_2)(\ell_1)$. That is, $\text{alt } \gamma = b^{-1}$ when b is regarded as the bicharacter $(\ell_1, \ell_2) \mapsto b(\ell_1)(\ell_2)$. These are the correspondences (4.3) of Section 4 with τ trivial.
- $gL^\vee \cap L^\vee = 0$, in which case $f(g)$ has a single simple object.

Lemma 5.1. *Fix σ as in (5.5). An element $g \in \mathcal{O}(A)$ satisfies $gL^\vee \cap L^\vee = 0$ if and only if $g = pT_\sigma p'$ for some $p, p' \in P$.*

Proof. If $g = pT_\sigma p'$ then $gL^\vee = pL$, and $pL \cap L^\vee = p(L \cap L^\vee) = 0$. Conversely, $K = gL^\vee$ is a Lagrangian transverse to $L^\vee$, so the projection $K \rightarrow A/L^\vee \simeq L$ is an isomorphism, and $K = \{(\ell, c(\ell))\}$ for some homomorphism $c: L \rightarrow L^\vee$. Since ev is trivial on K , c is alternating, so $K = S_c L$. Then $T_\sigma S_c^{-1}g$ sends $L^\vee$ to $T_\sigma L = L^\vee$, i.e. $p' := T_\sigma S_c^{-1}g \in P$, and $g = S_c T_\sigma p'$ since $T_\sigma^2 = 1$. $\square$

Corollary 5.1. *If $gL^\vee \cap L^\vee = 0$, then $f(g)$ is the composite of the images of a correspondence of Section 4, the Fourier kernel $C_{L,L}^\sigma$, and another correspondence of Section 4.*

Proof. This follows from Lemma 5.1, Theorem 5.1, the description of f on P above, and the fact that f is a homomorphism. $\square$

5.6.2. Fourier inversion.

Proposition 5.3. *Let $\sigma, \sigma': L \rightarrow L^\vee$ be symmetric isomorphisms as in (5.6), and let $u = \sigma'^{-1} \circ \sigma \in \text{Aut}(L)$. Then $T_{\sigma'} T_\sigma = D_u$, and there is an equivalence of $\mathbf{Vect}[L]$ -bimodule categories between the composite of the walls $\text{Sum}_3(C_{L,L}^\sigma)$ and $\text{Sum}_3(C_{L,L}^{\sigma'})$ (in this order) and $\text{Sum}_3(\Gamma(u))$, the image of the undecorated graph of u . In particular, for $\sigma = \sigma'$ the Fourier transform composed with itself is the identity.*

Proof. We have $T_{\sigma'} T_\sigma(\ell, \varphi) = (\sigma'^{-1} \sigma \ell, \sigma'(\sigma^{-1} \varphi))$, and evaluating on $k \in L$ and using (5.6) for σ and σ' gives $\sigma'(\sigma^{-1} \varphi)(k) = \sigma'(k)(\sigma^{-1} \varphi) = \varphi(\sigma^{-1} \sigma' k)$, so $T_{\sigma'} T_\sigma = D_u$. By Theorem 5.1, $\text{Sum}_3(C_{L,L}^\sigma) \simeq f(T_\sigma)$, and similarly for σ' . Since f is a homomorphism, the composite is $f(D_u)$, which is the image of the graph of u as above. $\square$

Remark 17 (Integrating out the middle variable). Proposition 5.3 can also be seen at the level of correspondences, as an analogue of the classical identity $\int e^{i(x-y)m} dm = \delta(x-y)$. The composite correspondence is $BL \leftarrow B(L^3) \rightarrow BL$, given by $(\ell, m, k) \mapsto \ell$ and $(\ell, m, k) \mapsto k$, decorated by $\kappa = p_{1,2}^* \kappa_L^\sigma \cdot p_{2,3}^* \kappa_L^{\sigma'}$. The map $L^3 \rightarrow L^2$ is not injective: its kernel is the “middle variable” $K = \{(0, m, 0)\}$. Suppose Sum_3 sends such a correspondence to the category of κ -twisted vector bundles on the homotopy fiber of $B(L^3) \rightarrow B(L^2)$, as expected from [FHLT10, §8]. Since $\text{alt } \kappa$ is trivial on K , this is $\mathbf{Rep}(K) \simeq \mathbf{Vect}[K^\vee]$, with $(\ell, k) \in L^2$ acting by translation by the character

$$m \mapsto \text{alt } \kappa((\ell, 0, k), (0, m, 0)) = \sigma(\ell)(m)^{-1} \sigma'(k)(m)$$

of K , computed using (5.9). (Replacing κ by κ^{-1} inverts this character and does not change its kernel, so the conclusion below does not depend on the sign convention for κ_L^σ .) The

stabilizer of a simple object is $\{(\ell, k) : \sigma(\ell) = \sigma'(k)\} = \Gamma(u)$, and, by (5.6), $\text{alt } \kappa$ is trivial on the subgroup of L^3 lying over $\Gamma(u)$. So the composite is the module category associated to $\Gamma(u)$ with trivial bicharacter: integrating over the middle variable collapses the two Fourier kernels to the graph of $u = \sigma'^{-1}\sigma$, and not of u^{-1} , in agreement with $T_{\sigma'}T_{\sigma} = D_u$ and the convention $f(gh) \simeq f(g) \circ f(h)$.

Remark 18. Suppose $L \simeq \mathbb{F}_p^n$ for an odd prime p . By Witt's theorem, two Lagrangians K, K' with $\dim(K \cap L^\vee) = \dim(K' \cap L^\vee)$ differ by an element of P , so $O(A) = \bigsqcup_{r=0}^n Pw_rP$. Here, for a decomposition $L = L_1 \oplus L_2$ with $\dim L_2 = r$, $w_r = T_{\sigma_1} \oplus \text{id}_{L_2 \oplus L_2^\vee}$ is a “partial Fourier transform”. The case $r = n$ is P itself, and the case $r = 0$ is Lemma 5.1. Correspondingly, every element of $O(A)$ acts via a composite of correspondences of Section 4 and the external product of a Fourier kernel for L_1 with the identity correspondence for L_2 . At the level of correspondences, w_r is related to the diagram in Section 5.1 for the non-transverse pair $(L, w_r L)$, where $L \cap w_r L = L_2$.

5.6.3. *Coherence and the anomaly.* Proposition 5.3 is the analogue of the composition law of the classical integral transforms. At this categorical level, however, the composition law is witnessed by an equivalence of bimodule categories rather than by a scalar. The equivalences

$$m(g, h) : f(h) \circ f(g) \simeq f(gh)$$

form a torsor over the group of isomorphism classes of invertible objects of $\mathcal{Z}(\mathbf{Vect}[L])$, i.e. over A . So the analogue of the scalar $c(g, h)$ is an A -valued 2-cochain on $O(A)$. As in Section 4, the obstruction to choosing these coherently is a class $O_3 \in H^3(O(A), A)$. Once a trivialization is chosen, the obstruction to choosing associators is a class $O_4 \in H^4(O(A), \mathbf{k}^\times)$, which is the anomaly of the projective action. For $|L|$ odd, O_3 vanishes and has a canonical trivialization [EG18, §6], which is the one used in Theorem 5.2.

Example 3. Let L be an elementary abelian p -group for an odd prime p . Then the class O_4 associated to the canonical trivialization of O_3 vanishes [EG18, Theorem 6.1] (see also [VD23]). Therefore the walls of this section and the correspondences of Section 4 can be completed to a coherent linear action of $O(A)$ on σ_{BL}^3 , and such completions form a torsor over $H^3(O(A), \mathbf{k}^\times)$. This is the analogue of the fact that the Weil representation of the symplectic group over a finite field of odd characteristic is linear [GH09, GH11]. It is expected that O_4 vanishes for all L of odd order [EG18, §6].

Remark 19. Let $O(L, \sigma) \subset \text{Aut}(L)$ denote the automorphisms u with $\sigma(u\ell)(uk) = \sigma(\ell)(k)$ for all ℓ, k . Then D_u commutes with T_σ for $u \in O(L, \sigma)$. Let $\chi_\sigma(\ell, k) = \sigma(k)(\ell)$ be the symmetric nondegenerate bicharacter giving the middle associator of $\text{Sum}_3(C_{L,L}^\sigma)$ (Proposition 5.2). Then $O(L, \sigma)$ acts by strict tensor automorphisms on the Tambara-Yamagami category $\text{TY}(L, \chi_\sigma, \epsilon)$ [TY98], for either sign $\epsilon = \pm |L|^{-1/2}$. The crossed product $\text{TY}(L, \chi_\sigma, \epsilon) \rtimes O(L, \sigma)$ is a graded extension of $\mathbf{Vect}[L]$ whose homogeneous components are the bimodule categories $f(h)$ for $h \in \langle T_\sigma \rangle \times O(L, \sigma)$. It gives an explicit linearization of the projective action restricted to this subgroup, though not necessarily the one associated to the canonical trivialization of O_3 .

Example 4. Let $L = \mathbb{Z}/p$ for an odd prime p , and identify A with $\mathbb{F}_p^2$ and ev with $(x, y) \mapsto xy$. The only Lagrangians are L and $L^\vee$, and

$$O(A) = \{D_a : a \in \mathbb{F}_p^\times\} \sqcup \{T_s : s \in \mathbb{F}_p^\times\},$$

where $D_a(x, y) = (ax, a^{-1}y)$ and $T_s(x, y) = (s^{-1}y, sx)$ is T_σ for the (necessarily symmetric) isomorphism $\sigma(x)(y) = sxy$. This is the dihedral group of order $2(p-1)$ [EG18, Proposition 3.6]. Since $H^2(\mathbb{Z}/p, \mathbf{k}^\times) = 0$, we have $P = \{D_a\}$, and the decomposition of Lemma 5.1 has

exactly two cells. The multiplication is

$$D_a D_{a'} = D_{aa'} , \quad T_{s'} T_s = D_{s/s'} , \quad D_a T_s = T_{s/a} , \quad T_s D_a = T_{sa} ,$$

and each relation is realized by correspondences: the first by composing graphs, the second by [Proposition 5.3](#), and the last two by composing a graph with a kernel, which has no fiber to integrate over and simply reparametrizes the kernel. Explicitly, write κ_s for the decoration of $C_{L,L}^\sigma$ with $\sigma(x)(y) = sxy$. Then $\Gamma(a)$ followed by $C_{L,L}^\sigma$ is L^2 decorated by $\kappa_s(al, k)$, which is κ_{sa} , matching $T_s D_a = T_{sa}$. Similarly, $C_{L,L}^\sigma$ followed by $\Gamma(a)$ is decorated by $\kappa_s(\ell, a^{-1}k) = \kappa_{s/a}$, matching $D_a T_s = T_{s/a}$. The elements T_s fall into two conjugacy classes, according to the class of s in $\mathbb{F}_p^\times / (\mathbb{F}_p^\times)^2$. By [Example 3](#), the anomaly vanishes, so this projective action of the dihedral group linearizes. On the subgroup $\{1, T_s, D_{-1}, T_{-s}\} \simeq \mathbb{Z}/2 \times \mathbb{Z}/2$, an explicit linearization is given by $\text{TY}(\mathbb{Z}/p, \chi_\sigma, \epsilon) \rtimes \mathbb{Z}/2$, with $\mathbb{Z}/2$ acting by inversion, as in the previous remark.

APPENDIX A. BICHARACTERS AND THE SCHUR MULTIPLIER

Let $\wedge^2(-)$ denote the group of alternating bicharacters, and let $Z^2(-, \mathbf{k}^\times)$ denote the group of degree two group cocycles. Let A be an abelian group. There is a map

$$\text{alt}: Z^2(A, \mathbf{k}^\times) \rightarrow \wedge^2(A)$$

given by sending a cocycle $\alpha \in Z^2(A)$ to the alternating bicharacter sending:

$$(a_1, a_2) \mapsto \alpha(a_1, a_2) \alpha(a_2, a_1)^{-1} .$$

It is a well-known fact that this induces an equivalence:

$$(A.1) \quad \text{alt}: H^2(A, \mathbf{k}^\times) \xrightarrow{\sim} \wedge^2 A .$$

See e.g. [NN08, 2.4].

If $A = A_1 \oplus A_2$ is an abelian group, recall the *Schur isomorphism*:

$$(A.2) \quad H^2(A) \simeq H^2(A_1) \oplus H^2(A_2) \oplus \text{Hom}(A_1 \otimes_{\mathbb{Z}} A_2, \mathbf{k}^\times) .$$

See e.g. [Kar87, Theorem 2.2.10].

There is an equivalence of $\mathbf{Vect}[A_1 \oplus A_2]$ -modules and $\mathbf{Vect}[A_1]$ - $\mathbf{Vect}[A_2]$ -bimodules sending a module to the same underlying category, with bimodule structure:

$$\mathbf{k}_\ell \otimes (-) \otimes \mathbf{k}_k = \mathbf{k}_{(\ell, -k)} \otimes (-)$$

where $\ell \in A_1$ and $k \in A_2$.

Lemma A.1. *Under this equivalence, $\mathbf{Vect}$ with module associativity constraint given by a 2-cocycle $\alpha \in Z^2(A_1 \oplus A_2)$ gets sent to $\mathbf{Vect}$ with middle associator*

$$(\ell, k) \mapsto (\text{alt } \alpha)((\ell, 0), (0, -k)) = (\text{alt } \alpha)((\ell, 0), (0, k))^{-1} .$$

Proof. Let $\alpha \in Z^2(A_1 \oplus A_2)$ be the associator for a $\mathbf{Vect}[A_1 \oplus A_2]$ -module category with underlying category $\mathbf{Vect}$. The middle associator for the associated $\mathbf{Vect}[A_1]$ - $\mathbf{Vect}[A_2]$ -bimodule category is an equivalence, for all $\ell \in A_1$ and $k \in A_2$ between

$$(\mathbf{k}_\ell \otimes -) \otimes \mathbf{k}_k = \mathbf{k}_{(0, -k)} \otimes (\mathbf{k}_{(\ell, 0)} \otimes -)$$

and

$$\mathbf{k}_\ell \otimes (- \otimes \mathbf{k}_k) = \mathbf{k}_{(\ell, 0)} \otimes (\mathbf{k}_{(0, -k)} \otimes -) .$$

Because A is abelian, this is provided by the following composition:

$$\begin{array}{c}
\mathbf{k}_{(0,-k)} \otimes (\mathbf{k}_{(\ell,0)} \otimes -) \\
\downarrow \alpha((0,-k),(\ell,0))^{-1} \\
(\mathbf{k}_{(0,-k)} \otimes \mathbf{k}_{(\ell,0)}) \otimes (-) \\
\parallel \\
(\mathbf{k}_{(\ell,0)} \otimes \mathbf{k}_{(0,-k)}) \otimes (-) \\
\downarrow \alpha((\ell,0),(0,-k)) \\
\mathbf{k}_{(\ell,0)} \otimes (\mathbf{k}_{(0,-k)} \otimes -)
\end{array}$$

The middle arrow is an equality since $A_1 \oplus A_2$ is abelian and $\mathbf{Vect}[A_1 \oplus A_2]$ has trivial associator. So the composite is the scalar $\alpha((\ell,0),(0,-k))\alpha((0,-k),(\ell,0))^{-1} = (\text{alt } \alpha)((\ell,0),(0,-k))$, and the second expression in the statement follows since $\text{alt } \alpha$ is a bicharacter. $\square$

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